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This result is due to Pontrjagin, Thom, Milnor, Averbuh, and Wall. (See [2, 9, 19].) For the definition of the Pontrjagin numbers $p_{i_1} \dots p_{i_n} [V] \in J$ the reader is referred to Hirzebruch [6]. These numbers are defined only if the dimension k is a multiple of 4.

The *oriented cobordism ring* $\Omega_* = H_*(\mathcal{D}_o)$ is defined as follows. For $V \in \mathcal{D}_o$ let $-V$ denote the same manifold V with the opposite orientation. We will say that

$$V \equiv V' \pmod{\partial \mathcal{D}_o}$$

if $(-V) + V'$ is the boundary of some manifold in \mathcal{D}_o . As an example, for any closed manifold V we have $V \equiv V \pmod{\partial \mathcal{D}_o}$ since

$$(-V) + V \approx \partial(V \times I)$$

where I denotes the unit interval. The set of all such congruence classes form the required group Ω_k . Again the cartesian product operation makes $\Omega_* = (\Omega_0, \Omega_1, \dots)$ into a graded ring.

It follows from Theorem 1' that Ω_k is a finitely generated group of the form

$$J \oplus \dots \oplus J \oplus J_2 \oplus \dots \oplus J_2$$

where infinite cyclic summands can occur only if $k \equiv 0 \pmod{4}$.

THEOREM 2'. — *The ring Ω_* , modulo the ideal consisting of 2-torsion elements, is a polynomial ring $J[Y_4, Y_8, Y_{12}, \dots]$ with one generator in each dimension divisible by 4.*

The complex projective space of real dimension $4m$ can be taken as generator for $m = 1, 2, 3$. However a different generator is needed in dimension 16.

For a description of the 2-torsion in Ω_* the reader is referred to Wall's paper.

2. MANIFOLDS WITH X-STRUCTURE.

In this section we will define the concept of an "X-structure" on the tangent bundle of a differentiable manifold; and study the corresponding cobordism theory.

First recall Steenrod's definition of a tensor field [15, § 6.4 and § 9.1 with mild alterations]. Every differentiable k -manifold V can be made Riemannian and hence has a tangent bundle with structural group O_k . Let X be any topological space on which the group O_k acts. Then we can form the weakly associated bundle with base space V and fibre X . This may be called the "tensor bundle of type X " and its cross-sections are "tensor fields". As an example, if $k = 2m$, then O_{2m} acts on the coset space O_{2m}/U_m .

A cross-section of the corresponding bundle is called a *quasi*-(or almost) *complex structure* on V . (See [15, § 41.10].)

We will modify this definition as follows, so that it makes sense for all dimensions simultaneously. Let O denote the union of the orthogonal groups $O_1 \subset O_2 \subset O_3 \subset \dots$ in the fine topology. Then we require that this infinite orthogonal group O act on the space X . It follows that each O_k acts on X . Hence there is a tensor bundle of type X over any manifold $V \in \mathcal{D}$.

Definition: A homotopy class of cross-sections of the tensor bundle with fibre X over V is called an X -structure on V . A manifold $V \in \mathcal{D}$ together with an X -structure on V is called an X -manifold. We will still use the single symbol V to denote this pair.

Now if V is an X -manifold then ∂V is also. Given any closed X -manifold V one can define a second X -manifold $-V$ so that

$$\partial(V \times I) \approx V + (-V).$$

Thus one can define a cobordism group for the class of X -manifolds. The resulting group will be denoted by $N_k(X)$ and called the X -cobordism group. (Following Atiyah [1] this could also be called the k -th "bordism group" of the O -space X .)

Example 1. Let O/U denote the union of the spaces

$$O_2/U_1 \subset O_4/U_2 \subset O_6/U_3 \subset \dots$$

in the fine topology with O acting on O/U in the usual way. Then a manifold with an O/U -structure will be called a *weakly complex manifold*. (Compare Hirzebruch [7].) For example

any complex manifold is quasi-complex and hence weakly complex. Any sphere can be given an O/U -structure although only S^2 and S^6 possess quasi-complex structures.

The following results are due to Milnor and Novikov.

THEOREM 1''. — *A closed weakly complex manifold V is the boundary of a weakly complex manifold if and only if its Chern numbers $c_{i_1} \dots c_{i_n}[V]$ are all zero.*

(Explanation: an O/U -structure on V determines a preferred U -bundle over V . Hence Chern classes are defined.) It follows that $N_k(O/U)$ is zero for k odd and is free abelian for k even.

THEOREM 2''. — *The graded group $N_*(O/U)$ has a natural ring structure, making it into a polynomial ring $J[Y_2, Y_4, Y_6, \dots]$ with one generator in each even dimension.*

As generators one can take certain algebraic varieties with their natural complex structures. (Compare [7]. It is not known whether connected varieties will suffice.)

Example 2. More generally one could use any subgroup G of the infinite orthogonal group in place of U . For example using the infinite symplectic group Sp we would obtain a cobordism ring $N_*(O/Sp)$ which is appropriate for the study of "weakly quaternionic manifolds". The following six groups seem particularly interesting:

$$1 \subset Sp \subset SU \subset U \subset SO \subset 0.$$

Starting from the right, the ring $N_*(O/O)$ is just the non-oriented cobordism ring N_* and $N_*(O/SO)$ is the oriented cobordism ring Ω_* . The rings $N_*(O/SU)$ and $N_*(O/Sp)$ are more or less unknown. (Compare the concluding remarks in [9].)

The ring $N_*(O/1) = N_*(O)$ has essentially been studied by Pontrjagin [11]. An O -structure on V is a trivialization of the tangent O -bundle of V (the "stable" tangent bundle). Manifolds which admit such a structure are called " π -manifolds". It turns out that $N_k(O)$ is isomorphic to the stable homotopy groups $\pi_{k+n}(S^n)$ of the n -sphere, with n large. This fact is the basis for Pontrjagin's method of studying homotopy groups.

Example 3. Let X be a space on which O operates trivially. Then an X -structure on V is just a preferred homotopy class of maps $V \rightarrow X$. As cases of particular interest X might be an Eilenberg-MacLane space or the classifying space of a group. How does one compute the groups $N_k(X)$?

The above definitions can be modified slightly by admitting only oriented manifolds. Thus one obtains groups $\Omega_k(X)$ where X is any space on which the rotation group SO acts. Again I do not know how to compute these groups. (Added in proof: See Conner and Floyd [21].)

Example 4. Let P denote the infinite real projective space, with the infinite rotation group SO acting in the natural way. The cobordism groups $\Omega_k(P)$ for oriented manifolds with P -structure can be called the *spinor cobordism groups*. This name is appropriate since a P -structure is roughly a "lifting" of the structural group of the tangent bundle to the infinite spinor group. A manifold admits a P -structure if and only if its Stiefel-Whitney class w_2 is zero. The groups $\Omega_k(P)$ have no odd torsion, but otherwise I do not know much about them.

3. MISCELLANEOUS COBORDISM THEORIES.

So far we have concentrated on differentiable manifolds. However one could equally well define a cobordism group based on the class \mathcal{T} of all compact topological manifolds. (Compare Brown [3, Theorem 3].) The natural correspondence $\mathcal{D} \rightarrow \mathcal{T}$ induces a homomorphism from the differentiable cobordism group $N_k = H_k(\mathcal{D})$ to the topological cobordism group $H_k(\mathcal{T})$.

Since Thom [16] has shown that Stiefel-Whitney classes can be defined topologically, we have:

THEOREM 3 (Thom). — *The homomorphism $N_k \rightarrow H_k(\mathcal{T})$ has kernel zero.*

Problem: Is this homomorphism onto?

Another possibility would be to consider the class \mathcal{C}_o of all compact, oriented, combinatorial manifolds. Whitehead [20] has shown that each differentiable manifold has a preferred class of triangulations. Hence there is a natural homomorphism from