AN EXTENSION OF A THEOREM OF DARBOUX

AN EXTENSION OF A THEOREM OF DARBOUX

by Dwight B. Goodner

(Reçu le 10 novembre 1960)

Darboux's intermediate value theorem for derivatives [1, p. 117] requires the existence of the derivative. Since the derivative may fail to exist, it seems desirable to have expressions which may serve us when there is no derivative. The purpose of this paper is to extend Darboux's theorem to functions which have finite one-sided " Dini derivates ".

Let f be a function defined on the closed interval $a \leqq X \leqq b$. We will denote the upper right, lower right, upper left, and lower left derivates of f [2, p. 188] on $a \leqq X \leqq b$ by $D^+ f$, $D_+ f$, $D^- f$, and $D_- f$, respectively. In addition, if p and q are real numbers, we will use

$$pD(+)f + qD(-)f$$

to denote any one of the relations

$$p_1 D^+ f + q_1 D^- f,$$
$$p_2 D^+ f + q_2 D_- f,$$
$$p_3 D_+ f + q_3 D^- f,$$
$$p_4 D_+ f + q_4 D_- f,$$

Theorem. If the function f is continuous on the closed interval $a \leqq X \leqq b$, if f has finite derivates at each point of the open interval $a < X < b$, and if k is a real number such that $D_+ f(a) < k < D^- f(b)$, then there exist numbers ξ, p, and q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, and $p + q = 1$ such that

$$pD(+)f(\xi) + qD(-)f(\xi) = k$$

Proof. To avoid unnecessary repetition we will prove the theorem for only one possible choice of derivates. The relations for other choices can be similarly proved.

Let the function F be defined on $a \leqq x \leqq b$ by $F(x) = f(x) - kx$. Then [2, p. 191]

$$D^- F(b) = D^- f(b) - k > 0 \text{ and } D_+ F(a) = D_+ f(a) - k < 0.$$

Hence there is a point ξ, $a < \xi < b$, such that F has a minimum at ξ. We note that $D^+ F(\xi)\, D^- F(\xi) \leqq 0$. If $D^+ F(\xi) = 0$, we choose $p = 1$ and $q = 0$. If $D^+F(\xi) \neq 0$, we choose

$$p = \frac{D^- F(\xi)}{D^- F(\xi) - D^+ F(\xi)} \text{ and } q = \frac{D^+ F(\xi)}{D^+ F(\xi) - D^- F(\xi)}.$$

In either case $p \geqq 0$, $q \geqq 0$, $p + q = 1$, and $p\, D^+F(\xi) + q\, D^-F(\xi) = 0$. Hence

$$p\left[D^+ f(\xi) - k\right] + q\left[D^- f(\xi) - k\right] = 0.$$

It follows that

$$p D^+ f(\xi) + q D^- f(\xi) = k$$

which was to be shown.

The reader will observe that we have not required the derivates at a and b to be finite. In fact, we need not require that all derivates be finite at each point of the open interval $a < x < b$. However, since these cases are easily resolved, they will be left for the consideration of the reader.

A closely related theorem may be obtained by substituting $D^+f(a) > k > D_- f(b)$ for $D_+ f(a) < k < D^- f(b)$ in the statement of our theorem. A proof of the new theorem can be obtained by marking minor modifications in the above proof.

REFERENCES

1. Goffman, C., *Real Functions*. New York, Rinehart and Company, 1953.
2. McShane, E. J., *Integration*. Princeton, Princeton University Press, 1944.

Department of Mathematics
The Florida State University
Tallahassee, U.S.A.