I. – INTUITION AND EXPERIMENT IN MATHEMATICAL TEACHING IN THE SECONDARY SCHOOLS

he has had a very thorough grounding in descriptive geometry. There is truly the risk pointed out by Professor Greenhill that after he has devoted a great deal of time to descriptive geometry he will be liable to shirk, or ignore, or neglect the more rigid mathematical method; but even if this undesirable consequence results (which in nowise appears necessary) I think that it may be taken as the lesser of two evils.

Briefly from a more general point of view I think that in the discussion of the circular there has been too much tendency to dwell on detail questions as to what should be taught and what should not be taught, and to ignore the principle that true education ought to be a process of developing and expanding the mind in those directions in which such development and expansion is most likely to serve the ends in view, and that the provision of a mental tool equipment is, or should be considered as merely an incidental.

TROISIÈME SÉANCE

Mardi 27 août, à 9 heures et demie du matin.

Présidence de M. R. Fujisawa (Tokio) et C. Godfrey (Osborne)

Ordre du jour :

I. — *Intuition and experiment in mathematical Teaching in the Secondary schools* (l'intuition et l'expérience dans l'enseignement mathématique des écoles moyennes), rapport présenté par M. D.-E. Smith (New-York). — Discussion.

II. — Remarques sur une biblographie de l'enseignement mathématique, par M. C. Goldziher (Budapesth).

III. — Prolongation du mandat de la Commission, Les travaux pendant la prochaine période.

I. — INTUITION AND EXPERIMENT IN MATHEMATICAL TEACHING IN THE SECONDARY SCHOOLS[1]

Report presented by

David Eugene SMITH (New-York)

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1. — Method of Investigation.

In the year 1911 the Central Committee appointed a subcommittee known as « Subcommittee A » charged with the duty of investigating the rôle of intuition in the teaching of

[1] The German topic as assigned was « Anschauung und Experiment im mathematischen Unterricht der höheren Schulen ». This was translated into French as « L'intuition et l'expérience dans l'enseignement mathématique des écoles moyennes ». The translation is not

secondary mathematics. It assigned to Dr. W. Lietzmann of Barmen, Germany, the work of preparing a questionnaire that should facilitate the work of securing the data upon which to base a report, which labour was diligently performed. The questionnaire was sent out during the winter of 1911-1912 and replies were received by April 1, 1912, from representative teachers of Austria, England, France, Germany, Switzerland, and the United States. These replies were held until July 1, and no others having been received by that date, to the writer of this paper was assigned the duty of preparing the present report.

In the several countries the method of investigation differed materially. In some cases the « rapporteur » sent out local questionnaires; in others he referred to the printed reports of the International Commission; while in others a committee considered the replies. The results seem to show that it was a matter of no particular significance which plan was adopted.

2. — Types of School Considered.

The questionnaire having been prepared in the German language, and naturally under the influence of the German school system, the terms used do not exactly meet the situation as it appears in all of the other countries. It is therefore necessary to define with some care the terms employed, and to give the English interpretation to the several questions.

The Gymnasien and « Realanstalten » of Austria and Germany corresponding rather closely in years to the lycées of France, and to the Gymnasien and « Kantonschulen » of Switzerland. They correspond somewhat less closely to the so-called « public schools » of England, while they are radically different from the public « high schools » of the United States. It therefore becomes necessary to define « Unterricht der höheren Schulen » as the teaching in those school years that correspond in the age of pupils with the school years of the Gymnasien, of the lycées, of the

exact, Anschauung being neither exactly Intuition nor exactly Perception, and neither Experiment or Experience being a satisfactory equivalent of the German word Experiment.

This report is based upon data collected by Dr. Walther LIETZMANN, Oberlehrer an der Oberrealschule in Barmen, Germany. The data were secured through replies to a questionnaire prepared by Dr. Lietzmann, these replies having been sent by the following gentlemen :

Austria, Prof. Dr. Erwin DINTZL, Vienna;

England, Charles GODFREY, M. A., headmaster of the Royal Naval College, Osborne;

France, M. Ch. BIOCHE, Professeur au lycée Louis-le-Grand, Paris;

Germany, The late Prof. Dr. P. TREUTLEIN, G. H. R., Direktor des Real- und Reform-Gymnasiums, Karlsruhe; and Dr W. LIETZMANN, Barmen.

Switzerland, M. H. FEHR, Professeur à l'Université de Genève, Geneva;

United States of America, Professors David Eugene SMITH, Teachers College, Columbia University, New-York City, and J. W. A. YOUNG, the University of Chicago, Chicago, Illinois.

« public schools » (in England), and of the last part of the « elementary school », all of the « high school », and the first two years of « college » in the United States. This means that the study includes the work of pupils from about the age of 10 to about the age of 19 years. The French translation of *écoles moyennes* seems best to describe the schools in question.

It should also be noted that the replies have reference to the work done in schools of a general type, including both the classical and the non-classical, and usually not to that done in such special schools as those devoted chiefly to agriculture, mechanical arts, navigation, mining, and the like. Austria, for example, expressly limits the report to the Gymnasium, Realgymnasium, Reformrealgymnasium, and Realschule. England considers what are described loosely as « public schools », those which supply the greater number of students to the ancient universities of Oxford and Cambridge, and other general secondary schools. Germany and Switzerland report only upon the work of the Gymnasien, Realgymnasien, and Oberrealschulen, and France chiefly upon that of the lycées and collèges. In the United States, owing to the fact that the school system differs materially from that of Europe, the report refers to the upper classes in the « elementary school », to all of the work of the general « high school », and to the first two years of the « college »[1]. In all the reports, however, emphasis has been laid upon the general type of school rather than the special.

3. — The Question of Elementary and Higher Schools.

It is evident that the scope of the inquiry might well have been enlarged so as to include the important question of intuition and experience in the Kindergarten, the Ecoles maternelles, the primary school, and even the home. As Dr. Montessori is now bringing into prominence in Italy and America, and as thousands of successful teachers everywhere have long been proving, intuition and experience play a very important part in the first stages of a child's education in general, and with respect to mathematics in particular. M. Laisant has brought this important question to the attention of his countrymen and Lietz and other well-known educators have done the same in other countries. It was felt by the committee, therefore, that it was better to call the attention of

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[1] The « elementary school » generally has 8 years or Grades, the child entering Grade I at about the age of 6-7, and finishing Grade VIII at the age of 13-14. The « high school » has 4 years, the pupil entering at the age of 14-15, and leaving at the age of 17-18. The « college » follows, the entering age being 17-18, and the 4 years being completed at the age of 21-22, the bachelor's degree being then conferred. The strictly-speaking university courses are entered at the age of 22-23, and the degree of doctor of philosophy may be obtained at the age of 25-26.

teachers to the importance of the question in the *écoles moyennes* at this time, and to reserve for further consideration the question of the primary schools. It is to be hoped that, in case the Central Committee is continued for another period of four years, this question will be considered with the care that it deserves.

As to higher institutions of learning it should be observed that the topic assigned to this committee is closely related to that assigned to Subcommittee B, *The Mathematical Training of the Physicist in the University,* and this relationship appears the more evident as we consider the report that has already been presented by Professor Runge at the session of August 26, 1912.

4. — Method followed in this Report.

It seems to the committee that it is advisable, in this report, to give a summary of the reports prepared by the representatives of the various countries, rather than to submit these reports in full. This is especially the case because the representatives of some of the countries have merely referred to pages of published monographs instead of giving the information directly. It also seems advisable to give this résumé under each of the general rubrics suggested by the committee rather than to attempt any other arrangement.

5. — The general situation.

Before proceeding to consider the questions in detail it is desirable to say a few words concerning the general situation in the various countries. The first thing that must strike any observer is that we are passing through a period of great change in all that pertains to secondary education, including the field assigned to this committee. A subject like descriptive geometry, for example, can hardly be said to occupy any definite position, so rapidly changing are the views concerning its importance, its significance, and the schools in which it should be taught. Austria makes much of it in the Realschule, less of it in the Realgymnasium, and almost nothing of it in the Gymnasium. This is what we might expect, but what descriptive geometry will do in the Reformrealgymnasium it is rather early to say, since this type of school is only beginning to appear in that country. In England there is seen the same spirit of unrest, and the training of boys in special lines is having its influence in the general fields as well. The teaching of boys who are going into the army, engineering, or surveying, tends to assume a more intuitional and experimental character than that of other boys, and the army requirements during the last 10 years have no doubt had a powerful reaction on the character of the teaching throughout the English public schools. On the other hand, as regards the teaching of boys other than specialists, probably the prevalent view in England is that the proper field for methods of intuition and experiment is in the middle and lower classes rather than the upper. These methods are viewed with suspicion by many masters who are concerned with the more able mathematical boys, and in many cases practical methods learnt by the boys in the lower classes are allowed to rest in the upper. Many teachers consent to the postponement of abstract methods during the earlier teaching only on the understanding that the abstract character of the higher teaching shall be preserved.

In the United States the spirit of unrest in all educational matters is also manifest, and in particular with respect to this whole question of intuition and experiment in mathematics. The question has been agitated for the past ten years, and many kinds of experiments have been tried, varying from an extreme laboratory method with a minimum of mathematics to the most abstract kind of work in which intuition and experiment played almost no part. It seems quite impossible to report as to any fixed policy or general consensus of opinion upon the subject.

We shall now take up *seriatim* the various topics set forth in the questionnaire that was sent out by Dr. Lietzmann last winter. Since it will not be profitable to present orally the report in full, the details being better assimilated from the printed page, a brief summary of the results will now be given.

In the work of measuring and estimating, a more practical form of mensuration seems to be developing, especially in Austria, Germany and Switzerland. England, France, and the United States seem to have given the matter less attention, or at least to have secured less definite results. An elementary trigonometry is more commonly found at an early period in the first three countries, thus allowing for outdoor work with simple instruments at an earlier stage.

In the matter of geometric drawing and graphic representation of solids, the various countries seem to be in a transition stage between the period in which this was considered part of the duties of the art teacher and that in which it is to be taken over by the department of mathematics. The tendency is general to consider this work as part of mathematics. The nature of the work, is not, however, at all settled; even the term « descriptive geometry » has no well-defined meaning. It may be said in a general way that the subject is taught in schools of the Oberrealschule type, but not in those of the Gymnasium type, and that the tendency is to increase the work both in amount and quality in the former.

Graphic methods of representing functions have become universal in the last generation. From the idea of a line representing an equation the tendency is at present to that of a graphic representation of a function. Just how much the pupil is acquiring of the function concept seems often to be questioned, and the whole subject is in the experimental stage at present. Of the value of squared millimeter paper there is no question anywhere, but it seems equally true that its use has been abused by the over-extensive treatment of equations and by its application to proving the obvious.

The contracted methods of computation that were prominently advocated fifty years ago do not seem to have advanced materially, owing to the feeling that they are not really practical. On the other hand the use of logarithms seems on the increase, and the slide rule has come into great favor in the technical schools where approximate calculation is prominent. Graphic methods of computation and of the approximation of roots of numerical higher equations (as worked out by Professor Runge, for example[1], have found but little place in the schools of the type under investigation. It is probably too early to say what their success will be in this kind of school, or when, if ever, any large body of teachers will be found capable of treating the subject.

In general it may be said that more progress towards the recognition of the rôle of intuition and experiment in secondary mathematics seems to have been made of late in Austria, Germany, and Switzerland than in England, France, and the United States[2]. A comparison of the work done in these several countries, particularly in the way of applying mathematics seriously to the problems of life, and of visualizing the work in mensuration, will be one of the valuable results of the present international movement.

The greatest questions of all, however, relate to the nature of geometry and to the treatment of the function concept. Other questions are important, but here is the dominant issue that must be met in the next few years.

The first of these questions is this : How much of the geometry of the secondary schools shall be inductive, and how much deductive? Few are ready to assert to-day that it is best to begin geometry with a study of Euclid or Legendre. There must be a preparatory stage, and this must be characterized by intuition and experiment. But how much time shall be assigned to this stage?

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[1] In his Columbia University (New-York) lectures.

[2] At least this is the deduction from the reports submitted in reply to the questionnaire.

and exactly what ground shall be covered? and, what is more important, to what extent shall intuition replace deduction of the Euclid type?[1] Must we have two or even three years of *Anschauungslehre,* as some have advocated, or is a year or a half year enough?[2] How much must the rigorist in geometry be compelled to concede to the demands of the non-mathematical mind? Was Newton right when he expressed the opinion that all this intuitive work is pretty, but that it is not geometry? Must the real geometry of the past go by the board, as went the mediaeval logic, or will its position be strengthened by this propaedeutic work? Is the value of the Euclidean type of geometry such as to save it from destruction, or is it to be so diluted by this consideration of pictures, of models, and of simple mensuration as to be unrecognizable? Is the over-powering force of to-day, the force of modern industry, the cause of the growth of intuition and experiment in geometry? And if so, what will the over-powering force of to-morrow, the force of social considerations, demand? These are questions that we hear about us, but they are questions which we are not yet able scientifically to answer, and fortunately they are not within the province of this committee to attempt to answer.

In general it may be said that it is the plan of the Teutonic countries to mix the intuitional and the deductive work from the outset, while in France, and now in England, the plan is to let an inductive cycle precede a deductive one. The United States is only beginning to talk about the question, whatever tendency there is being towards the Anglo-French plan. Now is either of these plans better than the other? and can this be proved? Or is the question one of racial habit? Would the German plan succeed in the United States, or is the French plan better adapted to such a conglomeration of races? Would the English scheme of arranging the work in three stages, with intuition and experiment the initial one, be better for Germany than the totally different scheme that characterizes most of the recent textbooks of that country? Many people look upon the German plan as unscientific, attributing its apparent success to the excellently trained teachers who carry it out, while others look upon the cycle arrangement as ultra-scientific. Which is right? or is neither right? It is questions of this kind that the reports of the Commission will help us to answer in the next few years; and for the present it suffices to state the problem and to call attention to its importance.

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[1] Readers are referred to the recent works by TREUTLEIN and TIMERDING, published by Teubner.

[2] See LIETZMANN, *Stoff und Methode des Raumlehreunterrichts in Deutschland,* 1912.

The second important question relates to the treatment of the function concept. Here the rôle of intuition, in the first steps, is more clearly defined, since we have no well-tried body of knowledge to be set aside. The chief argument for the elaboration of the function concept seems to be that the calculus has already found place in the schools under our consideration, and if it is to hold this place and continue to grow in strength, we must cease to impose it merely from above, — we must prepare for it from below. The notions of limit, variability, rate, function, and graph must be so gradually introduced and must become so clearly understood that when the calculus is reached they will be met as we meet familiar friends. How to do this economically is one of the problems relating to intuitional mathematics. It is one of the interesting facts of present education that teachers are demanding the elimination of the incommensurable quantity from elementary demonstrative geometry, only to find themselves face to face with a demand for the study of limits, functions, and rate of change. The movement in favor of the elaboration of the function concept, however, is too recent to judge of its permanence in secondary education. Starting in France within the last twenty years, and vigorously advocated in Germany within the last decade, it has much to commend it if reasonably treated [1].

6. — Measuring and Estimating [2].

Austria reports that much is now being made of this work in the lower and middle classes, and that this will have its influence in the higher classes. Some idea of the nature of this work may be obtained from the recently published *Praktisch-geometrische Schülerübungen für die unteren Klassen*, by Fr. Schiffner. As an illustration of the nature of this work Dr. Matter, of the Stiftsgymnasium in Seitenstetten, relates that the children in the second class (numbering from the lowest in the school) measure the height of a tower by the simple use of a measuring tape and a large protractor. They find two angles and the included side of a vertical triangle, and then lay this off on the horizontal plane, and finally draw it to scale and measure the resulting figure. Similar work is also done in the classes in geography. In order to ascertain how much work of this general nature is done in the upper classes of the Gymnasien, Realgymnasien, and Realschulen, Dr. Erwin Dintzl of Vienna, who reports for Austria, secured in-

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[1] See Schimmack, *Die Entwicklung der mathematischen Unterrichtsreform in Deutschland*, 1911.

[2] Messen und Schätzen (Mesure et estimation des grandeurs).

formation from 38 institutions. He finds that in Vienna there is difficulty in carrying out a scheme of field work in connection with trigonometry owing to a lack of proper instruments, the crowding of the school curriculum, and the proper demands of out-door play. Occasionally, however, facilities for this work are found, and he mentions a satisfactory Taschen-Universal-Instrument made by Neuhöfer for the use of students and at the price of 170 Kroner (say a little over £ 7, 178 fr., or $ 34.50). In other cities a satisfactory amount of geodetic work is being done in the upper classes, the proximity to the country making this feasible.

In England, 58 % of the « public schools » investigated by Mr. Godfrey attempt no work in practical geodetic measurements, but in the remaining 42 % such work is done, but as a rule only by special classes of students, such as those preparing to be surveyors, engineers, or army officers. Out of the schools reporting, 14 % have a theodolite, and others have plane tables or other instruments. In the others secondary schools 56 % do work of this kind. Apparently more work of this nature is done here than in the older type of « public school, » theodolites being found in 23 % of the schools, and simpler instruments in numerous others.

In France it is rare that a lycée or collège does any work in geodetic or astronomical measurement. In the *Ecoles des Arts et Métiers,* where the pupils are about 17 years of age, elaborate courses in surveying are given, and pupils become expert in the use of such instruments as the theodolite. This work is done in the department of mathematics. The teacher of geometric design is usually the teacher of mathematics, and he relates the work in surveying to that in drawing, as in the case of profiles, plots, and the like.

In Germany the work varies in the different states. Dr. Lietzmann states that in the Prussian schools the theodolite is usually found, and that along with it are seen simple instruments for angle measure, angle mirrors and prisms, measuring rods, and the like. Simple instruments are often made by the pupils, particularly instruments for the measure of angles[1].

Much is made of out-door work in the classes in geometry and trigonometry, in the measuring of heights and distances the pupils making use of the instruments[2]. In Mecklenburg and Oldenburg about $^2/_3$ of all the secondary schools give systematic

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[1] For the rôle of intuition in the classroom in geometry, and for the conduct of a class period, see Dr. W. Lietzmann, *Die Organisation des mathematischen Unterrichts an den höheren Knabenschulen in Preussen,* p. 65.

[2] Ib., p. 161. See also Dr. H. Wieleitner, *Der mathematische Unterricht... im Königreich Bayern,* p. 42, 62 ; Dr. E. Geck, *Der mathematische Unterricht... im Königreich Württemberg,* p. 25; Prof. H. Cramer, *Der mathematische Unterricht im Grossherzogtum Baden,* p. 35; Prof. J. Wirz, *Der mathematische Unterricht.. im Elsass-Lothringen,* p. 9.

field work, and even in the rest this work is not neglected. At least in certain parts of Germany apparatus and models for the teaching of geometry play some part, and a helpful list of materials is given in a recent monograph by THAER, GEUTHER, and BÖTTGER [1], and in one by CRAMER [2]. The testimony of teachers there as elsewhere, however, is that there is a danger in the extensive use of models in geometry, although a reasonable use of apparatus in field work has a value that cannot be doubted.

With respect to mathematico-astronomical work in the German schools, it is coming to be recognized that such work is feasible and desirable. Numbers of Höherenschulen in Prussia have telescopes for astronomical observation although those that are suited to measuring are not usually found [3]. The subject has recently been treated in a monograph by Professor Hoffmann, and to this article reference must be made for detailed information [4]. The extent to which he has carried this work in his school is indicative of the German *Lehrfreiheit,* a freedom that is wanting in free America and in most other countries.

The helpful table that Dr. BRANDENBERGER has prepared to show at a glance the work done in the Gymnasien and Realschulen of Switzerland [5] gives the impression that relatively little field work in geometry, and even less in astronomy, is done in the Gymnasien. In Kanton Bern surveying is given in the summer; in Unterwalden some practical work is done with the theodolite and in the measuring of lengths and areas; a few Gymnasien give an hour a week to surveying during one year, usually with pupils of the age of 16-17 years; a few others possess observatories with sufficient apparatus for mathematical work. Of 25 Realschulen reporting, however, 12 give surveying as a definite topic in their curricula [6] and the rest make more or less of the subject in connection with trigonometry. A number of the schools are supplied with such instruments as the theodolite, cross staff, angle mirror, angle prism, and plane table. In certain schools, as in the technical classes at Lugano and St. Gall, very satisfactory courses are given including triangulation, the measurement of base lines, the making of profile maps, and the finding of altitudes.

In the United States there are, generally speaking, no required courses in trigonometry or geodesy in the « high schools », — that

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1 THAER, GEUTHER, und BÖTTGER, *Der mathematische Unterricht in den Gymnasien... Mecklenburgs und Oldenburgs,* pp. 22, 24, 27, 78.

2 *Der mathematische Unterricht im Grossherzogtum Baden,* p. 30.

3 LIETZMANN, *Die Organisation,* p. 43.

4 Prof. B. HOFFMANN, *Mathematische Himmelskunde und niedere Geodäsie an den höheren Schulen.*

5 Dr. K. BRANDENBERGER, *Der mathematische Unterricht an den Schweizerischen Gymnasien und Realschulen,* pp. 13-25; p. 57.

6 Ib., pp. 60, 62, 119.

is in those 4-year schools that are intermediate between the 8-year elementary schools and the 4-year college. In the third or fourth year when the pupil is 16-17 years old, elective work is offered in trigonometry by the better class of high schools. Occasionally the theodolite or some similar instrument is used, but in general the « high schools » do not possess such apparatus. Such work is regularly taken up in the first year in college, and then as an elective study. It may thenceforth be carried on as an elective by students of general mathematics, but it is required of engineers.

In certain special schools, however, some interesting work may be found. For example, Principal Stark studied with the writer not long ago the subject of primitive mathematical instruments and their relation to present-day teaching. The results were put in practice in the Ethical Culture School in New-York, the children making astrolabes from paper protractors and deriving much interest from their out-door measurements with these and other instruments. In the Horace Mann School, connected with Teachers College, New-York, there is at present a class in vocational mathematics, well supplied with simple instruments which they use out of doors in their eighth school year, preparatory to the more scientific work that follows[1].

Astronomical work requiring the use of instruments may be begun in the first year of college and may be continued thereafter. It is elective throughout. The student who pursues the courses will be prepared to do scientific photographic work in the senior year (the fourth year in college, about 21-22 years of age). Colleges of the better class are supplied with material for work of this character.

The educational problem may be stated generally as follows: What simple, inexpensive instruments may advantageously be used to increase the interest in the early stages of mathematics? In particular, what can be done to make the inductive cycle or phase of geometry more real and interesting, without weakening the deductive side? Then what further apparatus can be used to advantage in the later geometry and trigonometry?

7. — Geometric Drawing and Graphic Representation[2].

The questionnaire asked first concerning the work done in descriptive geometry (oblique parallel projection), preparation of plans and elevations, central projection, and the theory of shad-

[1] The class is conducted by Mr. J. C. Brown who has contributed to the psychology of elementary drill work in mathematics.

[2] Zeichnen und Darstellen (Dessin et représentation graphique).

ows; concerning the fusion of this teaching and the teaching of stereometry; and concerning the teaching of the subject in the department of mathematics or the department of drawing.

In Austria, in the Realschulen, the subject is nominally in the hands of a special teacher examined and appointed for the purpose. Professor Dintzl raises the question as to the wisdom of this arrangement, saying that to the qualifications of a teacher of descriptive geometry must be joined those of a teacher of mathematics if we are to have successful results. At present in about a third of the schools the teacher of mathematics is also the teacher of descriptive geometry in the three upper classes (V-VII). There is a well-defined line of demarcation between stereometry and descriptive geometry as the work is carried on, although the latter is employed in representing the solids met in the former. It appears, therefore, that certain definite work in descriptive geometry is done in the upper classes of the Realschulen. In the Gymnasien descriptive geometry is not a separate object of study. In the work in stereometry, however, plans and elevations are drawn, and orthogonal projections of parallelepipeds, octahedrons, pyramids, and the like are prepared [1].

In England descriptive geometry was formerly taught only to boys who needed it for special examinations, and then generally by the art master, the result being mere work by rule with great attention to artistic ink-work. Two or three causes are operating to change this position: (1) The Army examiners now treat the subject as part of mathematics, and do not require inking-in, the result being that mathematical masters are not discouraged from teaching the subject by consciousness of their technical weakness on the artistic side. (2) Descriptive geometry is now required for an honours degree in mathematics at Cambridge, and this circumstance will result in an increased output of mathematical masters interested in the subject and will have its direct reaction on school work. (3) Boys intending to study engineering at the University are encouraged to master descriptive geometry and perspective at school. Oblique parallel perspective is little studied in English schools. Descriptive geometry according to Monge (with work in plans, elevations, rabattements, etc.) is taught in 77 % of the « public schools » reporting, and perhaps in half of these it now forms part of the school course for the upper classes, in other cases being confined to specialists. Of the other secondary schools, 60 % have it in their courses. It seems not to have established itself, however, because it does not enter into

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[1] See also ADLER, *Der Unterricht in der darstellende Geometrie an den Realschulen, Gymnasien, Realgymnasien und Reformgymnasien,* and MÜLLER, *Der Unterricht in der darstellenden Geometrie in den technischen Hochschulen,* in Heft 9 of the Austrian reports.

the ordinary school examinations. Where it is taught two thirds of schools « correlate it with solid geometry, and in the majority of the « public schools » cases it is taught by a mathematician. It is seldom that the « public schools » do what is often done in other secondary schools, namely correlate descriptive geometry with manual work in the carpenter's shop. Perspective, being more technical than descriptive geometry, has not been so extensively taken over by the department of mathematics. In 63 % of the « public schools » reporting, it is taught in connection with drawing and usually by the drawing teacher. Accurate drawing in connection with projective geometry, including the construction of conics by anharmonic (cross) ratio properties, is not done at all in 69 % of the schools, and only a few of the most advanced boys in the other schools are encouraged to reach this comparatively advanced work.

France reports that descriptive geometry is taught in the lycées and collèges only in 1^re^ C and D (the pupils being 16 years of age), in « Mathématique A et B, » and in the classes preparing for the higher scientific schools[1].

The teaching of descriptive geometry is generally distinct from that of stereometry. Usually geometric drawing is not taught by the teacher of mathematics, but at present there is an active effort being made to put the direction of the work in the department of mathematics[2].

Germany, like Austria, seems to be taking a leading position in this line of work. Models for the study of descriptive geometry seem to be not uncommon. In the Realgymnasien and Oberrealschulen especially there is given definite work in this line. Oblique parallel projection is taught in Prussia in the Obersekunda (the seventh year of this type of school, or the pupil's tenth school year), and descriptive geometry as a special subject is taught in Oberprima (the last, ninth year[3]). The tendency in Northern Germany has been to make descriptive geometry in its various phases an elective subject in the domain of art instruction. In Saxony and the Southern states it has more generally been required[4]. The tendency throughout Germany seems to be in favor of putting geometric drawing, under whatever name, in the hands of the teachers of mathematics[5]. In South

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[1] See the report of M. ROUSSEAU and that of M. BLUTEL (vol. II).

[2] The report as sent to this committee was very brief, and for further information the reader is, therefore, referred to the printed reports of the Commission.

[3] See particularly, LIETZMANN, *Die Organisation*, etc., p. 139; and ZÜHLKE, *Der Unterricht im Linearzeichnen und in der darstellenden Geometrie an den deutschen Realanstalten*.

[4] THAER, GEUTHER, und BÖTTGER, *loc. cit.*, p. 81.

[5] Ibid., pp. 31, 32, 36, 56, 70, 73; WIELEITNER, *loc. cit.*, pp. 29, 35, 43, 56; WITTING, *loc. cit.*, pp. 27, 29, 33, 36, 57, 64; CRAMER, *loc. cit.*, pp. 15, 20; SCHIMMACK, *loc. cit.*, pp. 13, 15, 20, 43; WIRZ, *loc. cit.*, pp. 9, 17, 25, 31, 50; ZÜHLKE, *loc. cit.*

Germany the work has for a long time been in a satisfactory condition, but it was quite neglected in North Germany until 1898. In general it may be said that geometric drawing is commonly given in the Gymnasien and Oberrealschulen; that, as might be expected, it is found chiefly in Sekunda and Prima; that in the Oberrealschulen it has much more attention than in the Gymnasien; that it was formerly related to the art work but now it is being taken over by the mathematical department; that in the non-technical schools the descriptive geometry is related more to the representation of the usual geometric solids by projection than directly to the making of the working drawings of the artisan; and that more attention is being paid to the subject than is the case in the non-technical schools of any other country except Austria and Switzerland.

In Switzerland[1] the Gymnasien devote little attention to descriptive geometry, most of them not mentioning it[2]. In the Realanstalten, however, it is found almost without exception in the last three years[3]. In the published curricula Switzerland makes very clear the nature of the work that is done under the general terms of « Geometrisches Zeichnen » and « Darstellende Geometrie. » For example, in the curriculum of Bern the course includes work in geometric ornament, such as parquetry flooring; orthogonal projection of geometric solids; drawing of conics and other plane curves; drawing of machine models; shadow constructions; axonometry; polar perspective; solids of rotation; and plans and elevations. Such work, in a non-technical school, is found in the reports of only the three countries above named, among those that have replied to the questionnaire. Moreover it appears that in the large majority of the schools the practical bearing of all of this work upon industry is fully recognized. It also appears that the work in descriptive geometry is looked upon as belonging to the teacher of mathematics rather than to the teacher of art[4].

In the United States few general « high schools » offer work in geometric drawing of any kind. Formerly some such work was occasionally given by the teacher of drawing, but the awakening of an interest in art a few years ago led to the substitution of more free-hand drawing and color work for the mechanical drawing then somewhat in vogue. The change was justified, for the mechanical drawing was badly done as a rule and the country needed all of the improvement in art appreciation that it could

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[1] See Brandenberger, *loc. cit.*, pp. 14-25, 129-137.
[2] Ibid., pp. 14-20, 129.
[3] Ibid., pp. 20-25, 130. One of the clearest statements of the nature of the work to be found in any of the reports appears on pp. 130-135.
[4] Ibid., pp. 60, 62, 118, 137.

get. In the technical « high schools » that are beginning to appear in the larger cities, work of this kind is being begun, but not as yet with the thoroughness that characterizes the Austrian, German, and Swiss schools. In the technical courses of the colleges it is begun in the freshman year (about 18 years of age) and is taken only by students of architecture or engineering. These courses are rarely given by the mathematician, being left to the engineer or architect.

The educational problem may, therefore, be stated as follows : Are we making enough of drawing in connection with geometry? In particular, are we recognizing the practical value, not merely of the working drawings of the artisan, but also of geometric design, of cartography, of topographical drawing, and of geometric construction?

8. — Graphic Methods [1].

Graphic methods of one form or another are now found in the courses in mathematics, at least in the Realanstalten, in all countries, having gradually made their way from engineering, through thermodynamics and general physics, to pure mathematics. The extent to which these methods are used varies, however.

In Austria millimeter paper is used for all of the simple functions, such as the elementary ones of algebra, and the exponential, logarithmic, and circular functions. The value of the sine is found to 2 decimal places by measuring on the squared paper, and computations are often verified graphically. The planimeter is rarely used. Vector geometry is used for the purpose of representing the complex number, and for elucidating certain parts of mechanics. For the work in complex numbers it appears in the sixth class (reviewed in the eighth), and for mechanics it is introduced in the sixth (Realschule) or seventh (Gymnasium) class. The real consideration of the scalar field is hardly undertaken, but it is touched upon in the study of magnetic and electric fields, the concept of the potential, and the like, in the upper classes.

In England about 90 % of the schools state that the graphical study of statistics is given. It seems to begin mainly in the lower and middle classes of the schools. Probably it involves little beyond the plotting of tables of statistics, and for a new subject it has

[1] Graphische Methoden. (Darstellung von Funktionen auf Millimeterpapier. Vektorendarstellung. Skalare Felder. Graphisches Rechnen, insbesondere graphische Statik. Flächenauswertung mit Millimeterpapier oder Planimeter).

Les méthodes graphiques. (Représentation de fonctions sur du papier millimétrique. Représentation des vecteurs. Champ scalaire. Calcul graphique et spécialement statique graphique. Evaluation de surfaces à l'aide du papier millimétrique ou du planimètre.)

made rapid progress, having both the encouragement of the mathematicians and an abundant opportunity for application. The graphical representation of functions is taught in all of the « public schools ». In 27 % of the schools it is begun in the lower classes, in 58 % in the middle classes, and in 15 % in the upper classes. Similar results are found in the other secondary schools. The work is connected with the plotting of equations and with the approximation of the roots. Whether the idea of functionality is really grasped in this work seems open to question, but the graph forms a basis from which further advance may be made. The use of vectors is found in a large majority of the schools, in connection with mechanics (velocities, accelerations, forces), this latter subject being part of the mathematical course in England. In some schools the vector is used in connection with complex numbers. Graphical statics is taught generally. Areas are estimated by squared paper in most schools, but the planimeter is rarely used.

In France the notion of coordinates is introduced when the pupil is about 14 years of age. The graph is used in the study of equations, as apparently in all other countries. The technical schools make use of graphic mechanics. In the third year of the Ecole des Arts et Métiers (age of pupil, 20 years) and in the higher scientific schools graphical statics is taught. The planimeter is not used.

In Germany the custom is growing of having, at least in the class-rooms of Tertia or Sekunda, a portion of the blackboard ruled in squares, usually about 5 cm. on a side, for the graphic representation of functions [1]. In some of the Gymnasien [2] there is carefully organized work in such representation of functions from Obertertia on. In Obertertia such functions as

$$y = \frac{b}{a}\sqrt{a^2 - x^2}\,, \quad y = \frac{1}{x}\,, \quad y = \sqrt{a^2 + x^2}\,, \quad \text{and} \quad y = ax^3 \quad \text{or} \quad ax^4$$

are treated graphically. In Untersekunda this is extended to the exponential, logarithmic, and trigonometric functions [3], and in the Obersekunda and Prima to functions of degree exceeding two [4]. In all cases the function concept is mentioned as prominent, and the use of the graph merely for purposes of solving an

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[1] LIETZMANN, *Die Organisation*, p. 37.
[2] Ibid., p. 161.
[3] Ibid., pp. 161, 170.
[4] See also THAER, GEUTHER, und BÖTTGER, *loc. cit.*, pp. 6, 9, 79; WIELEITNER, *loc. cit.*, pp. 20, 42; WITTING, *loc. cit.*, p. 36; GECK, *Der mathematische Unterricht... im Königreich Württemberg*, pp. 21, 25, 59; SCHNELL, *Der mathematische Unterricht... im Grossherzogtum Hessen*, pp. 20, 29, 39, 41.

equation is not in evidence[1]. Graphic aids in computation are also employed to some extent in the German schools[2].

In Switzerland the graphic representation of equations and functions is general, as in other countries[3], and is extended to the treatment of limits. The question of the function concept is not settled there, and it does not appear to be settled anywhere. Exactly what can be done, and how it is to be begun, are matters still awaiting the results of experience. Dr. Brandenberger calls attention, for example, to the fact that in the course of study of one Realschule (where it would be expected) the word does not appear at all, while in a certain Gymnasium (where mathematics is less in evidence) it plays a prominent part. It would seem that the practice does not differ so much as the printed statements, and that the notion of function is introduced generally when there is a demand for it. Some schools emphasize it early, others late, and experience seems not as yet to have given any definite verdict.

In the United States the graphic representation of simple algebraic functions, with millimeter paper, is commonly begun in the first year of the « high schools » (about 14 years of age), and continued, chiefly with curves of the second degree, in the third year (age 16 years). The graphic treatment of vectors and scalars is reserved for the college, where it commonly begins in the algebra work of the freshman year (age 18 years). It is not taken up with any thoroughness unless the student elects a course in vector analysis, which is frequently offered in the junior or senior year (age 20, 21 years), and is frequently relegated to the graduate years (about 22-24 years).

9. — **Numerical Computation**[4].

The inquiry of the committee related to abridged computation, to the use of the slide rule and tables, and to graphical and numerical methods of approximation of the roots of equations.

In Austria abridged methods of computation are introduced in the third class (age about 13) in connection with the mensuration of surfaces[5]. The slide rule has not yet found general acceptance in the secondary schools. The reason for this seems to be the

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[1] See particularly Schimmack, *loc. cit.*, pp. 19, 22, and Wirz, *loc. cit.*, p. 45.
[2] Timerding, *Die Kaufmännischen Aufgaben*, p. 35.
[3] Brandenberger, *loc. cit.*, pp. 95, 101, 103, 104-109.
[4] Rechnen und Berechnen. Calculs et évaluations numériques.
[5] See the Austrian reports, Heft 3, S. 40, and Heft 6, S. 9.

expense of the instrument, the cheaper ones not being accurate enough to be of value; but the question of time to acquire the necessary facility is also a serious one. In the upper classes the numerical computation is performed almost exclusively by the aid of logarithms. The new programme requires the elements of probabilities in the upper class, and the simple problems of life insurance will require the use of mortality tables.

In England the mechanical rules for contracted operations with decimals have long been in use, but it seems that at present these formal rules are losing ground. The omission of useless figures and the presentation of results to a given number of significant figures must become increasingly general as practical and laboratory work in the lower classes gains ground. On the other hand many masters say that logarithms prove more useful than contracted methods, and that contracted rules are learnt but are seldom used afterwards. In 30 % of the « public schools » and in 66 % of the other secondary schools the slide rule is not used; it is generally used by pupils preparing for special examinations. The common use of the 4-figure logarithmic table is doubtless the cause for the little use of the slide rule. Tables of squares and square roots are not used in 40 % of the « public schools », and tables of cube roots are not used in 55 % of the « public schools ». The other secondary schools make less use of them. Tables of logarithms are used in all schools, 48 % introducing them in the upper classes and 37 % in the middle classes. Tables of trigonometric functions are used in all schools, in about half of them being introduced in the upper classes and in about 20 on 25 in the middle classes. Mortality tables are not generally used. In about 65 % of the schools 4-figure tables are used. Graphical methods of treating transcendental equations are not introduced systematically, and are found in 47 % of the « public schools » and in 16 % of the other secondary schools.

In France the abridged methods that became prominent in the middle of the 19th century are no longer taught. They seem not to be demanded in practical work. The slide rule is not used in the lycées and collèges except in classes preparing for technical schools. In the scientific classes of the lycées and technical schools the pupils learn the use of logarithms, usually with tables to five decimals. Trigonometric tables, with the logarithms of the functions, are common, as are tables[1] of square and cube roots and of log $(1 + r)$. The approximate solution of numerical higher equations, either graphically or by numerical computation, is given only in the *classes de mathématiques spéciales*.

[1] For example, Combet's tables, published by Belin, Paris.

In Germany contracted operations (abgekürzts Rechnen) are not required in the Prussian programme, although they are given in particular institutions[1]. The question of the real value of the work seems unsettled. In Bavaria it has some place in the lower classes of the Gymnasium and Realschule[2]. In Saxony the particular need for the subject is found in the Obertertia or Untersekunda of the Gymnasium[3]. In Baden and Elsass-Lothringen it is assigned place in Quarta of the Oberrealschule[4], and in general it may be said to have at least a nominal position in the various states of the Empire. The slide rule seems to be making its way very slowly in Germany as elsewhere[5]. The use of tables is general there as in other countries.

Five-place tables are used in the majority of schools, but about a third use four-place tables, interpolations being made mentally instead of by the use of proportional parts. In the lower classes the method of computing the tables is mentioned incidentally, the exercises in actual computation by means of series being relegated to the higher classes of the Oberrealschulen. The graphic treatment and the numerical approximations of the roots of numerical higher equations has place in the modern reform movement[6]. In the Realanstalten, however, the use of approximate methods (such as Newton's rule and the *Regula falsi*) is found in Oberprima.

In Switzerland the contracted operations are emphasized in the technical schools. The slide rule is found in the programmes of 6 of the 25 Realschulen, and in 2 Gymnasien[7]. Logarithms are universal, the majority of schools using 5-place tables, although about 30 $^0/_0$ use 7-place tables. Some advance is being made in the direction of decimalizing the angle, tables having been prepared for such divisions. Tables of roots and mortality are in use. In 25 $^0/_0$ of the Gymnasien the subject of infinite series is introduced, and applications are made to the theory of computing tables of logarithms. Methods of approximation of the roots of numerical higher equations are given. Graphic methods, the *Regula falsi*, and the Newtonian approximation are given in all Realschulen, and certain schools add Horner's, Lagrange's, or Gräffe's method.

In the United States contracted methods with decimals are

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1 Lietzmann, *Die Organisation*, pp. 104, 159; *Stoff und Methode... Unterricht*, p. 85.
2 Wieleitner, *loc. cit.*, pp. 24, 41.
3 Witting, *loc. cit.*, p. 16.
4 Cramer, *loc. cit.*, p. 18; Wirz, *loc. cit.*, pp. 29, 33.
5 Lietzmann, *Stoff und Methode... Unterricht*, p. 70 seq.; Timerding, *Die kaufmännischen Aufgaben*, p. 35.
6 Lietzmann, *Die Organisation*, pp. 161, 173, 178, 182, 197; Thaer, *loc. cit.*, p. 73; Wieleitner *loc. cit.*, p. 31; Witting, *loc. cit.*, p. 24.
7 Brandenberger, *loc. cit.*, pp. 60, 90.

rarely taught, either in the elementary schools or the general « high schools ». In the « technical high schools » (a relatively new type, parallel to the older type of general « high school »), and in the colleges where engineering courses are given, these methods are taught incidentally in the courses where they would naturally be used. The slide rule is coming into extensive use in the « technical high schools », and it is used in all engineering courses in the colleges. In the better types of general high schools it is occasionally explained to the classes in trigonometry. Mathematical tables are not used in the « high schools » until trigonometry, which is an elective study, is reached. In general, 5-place tables are used. In some arithmetics and in some official courses the table of compound interest is used in the eighth grade (age 13 years). Mortality tables are rarely seen in the secondary school. The method of computing tables is not taken up in the elementary classes, but in the calculus and in higher algebra (college studies, age 18-19) the student is shown how certain functions ($\sin x$, e^x , $\log x$, etc.) are expanded in series and how these series may be used in the making of tables. Graphic and arithmetic solutions of numerical higher equations are given in the college course in algebra (age about 18 years), the arithmetic solutions being effected usually by Horner's method. Graphic methods of solving such equations are beginning to be more highly appreciated.

The bearing of this type of computation upon the industrial problems of the present is important. In our time the typewriter has replaced the pen to a great extent, and the computing cash register has replaced the old cash drawer. Japan has kept her soroban [1] because she believes that the era of mechanical calculation is to return. In the Western World the cheapening of the calculating machine is testifying to Japan's forethought. What should the schools do in view of this change? Already the attempt to train the « lightning calculator » is past, but are we entering upon a period of practical graphical computation? of the replacing of logarithmic tables by the slide rule? and of inexpensive machines that shall perform the ordinary calculations, as they already perform the extraordinary ones? If so, are we to make the mistake that we made with logarithms, of using 7-place tables when 4-place ones would be better, or can we select with scientific forethought the material that is really practical?

This committee was charged with the duty of reporting upon present conditions and of propounding questions, but not of making recommendations as to the future. Since the stating of a

[1] See the report of Professor Fujisawa.

problem is quite as important as the solution, it is hoped that some of the questions that have been raised will serve a good purpose. That certain of these problems deserve more than a passing notice need hardly be asserted[1].

Questionnaire. — Rappelons ici le questionnaire qui a servi de base à l'étude de M. Smith. Il avait été rédigé et distribué par M. Lietzmann, au nom de la Sous-commission A.

Extrait de la circulaire de M. le D[r] W. Lietzmann. — Objet : *L'intuition et l'expérience dans l'enseignement mathématique des écoles moyennes.*

Délimitation du sujet. L'intuition et l'expérience jouent un rôle prépondérant dans l'enseignement géométrique des écoles élémentaires et des cours complémentaires (Fortbildungschule), ainsi que dans l'enseignement propédeutique des écoles moyennes. Dans la suite, il ne sera pas question de tout cela. Nous allons de même laisser de côté les cas multiples où l'intuition et l'expérience sont destinées à compléter ou à remplacer les développements logiques et déductif, dans l'enseignement systématique de la géométrie dans le domaine des éléments d'Euclide. Nous aurons peut-être l'occasion, dans une séance ultérieure de la Commission, d'examiner ces questions extrêmement importantes, en tenant compte du point de vue psychologique. Afin de bien délimiter le sujet, il conviendra donc à Cambridge de s'en tenir au rôle de l'intuition et de l'expérience dans les classes supérieures des écoles moyennes.

Pour organiser les travaux préparatoires, il est désirable d'avoir un tableau de l'état actuel de ce qui se fait dans les différents pays. Nous nous permettons à cet effet de vous soumettre les questions suivantes :

1. *Mesure et estimation des grandeurs.* Dans quels établissements, gymnase, école réale supérieure, etc., dans quelle étendue et dans quelles classes (âge des élèves).

a) Procède-t-on à des mesures *géodésiques* pratiques pour les utiliser ensuite numériquement ?

Usage du théodolite, de la chaîne d'arpenteur, etc.

b) Fait-on des observations et des mesures *astronomiques* avec des problèmes qui s'y rattachent ?

Usages d'appareils photographiques, instruments universels.

2. *Dessin et représentation graphique.* Dans quels établissements, dans quelle étendue et dans quelles classes présente-t-on :

a) La géométrie descriptive (Projection oblique ? — Plan et élévation ? — Projection centrale ? — Théorie des ombres ?)

Y a-t-il fusion entre cet enseignement et l'enseignement de la stéréométrie ? L'enseignement est-il donné par le maître de mathématiques ou par le maître de dessin ?

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[1] After this report was ready for the press the note by Messers. Cardinaal and Barrow, relating to the schools of Holland, appeared in *L'Enseignement mathématique*, July, 1912, p. 327. The plea for the freedom of the teacher in the matters referred to in this report is worthy of careful consideration.

b) Les méthodes graphiques (Représentation de fonctions sur du papier millimétrique ? — Représentation des vecteurs ? — Champ scalaire ? — Calcul graphique et spécialement statique graphique ? — Evaluation de surfaces à l'aide du papier millimétrique ou du planimètre ?)

3. *Calculs et évaluations numériques.*

a) Calcul abrégé à l'aide de fractions décimales ?

b) Emploi de la règle à calcul ?

c) Tables numériques (nombre de décimales pour le calcul logarithmique et les fonctions trigonométriques ? — Emploie-t-on aussi des tables de racines carrées ou de racines cubiques et des tables de mortalité ? — Est-ce que l'on montre à l'aide d'exemples comment on peut calculer les valeurs des logarithmes et des fonctions trigonométriques ?)

d) Résolution numérique et graphique des équations par approximation (Règle de Newton ? — Regula Falsi ? — Méthodes nomographiques ?).

Les deux publications ci-après permettent d'orienter le lecteur sur quelques-unes de ces questions et sur les réponses concernant l'Allemagne :

P. Zühlke : *Der Unterricht im Linearzeichnen und in der darstellenden Geometrie* (III. Bd., Heft 3 der Abhandl.).

B. Hoffmann : *Astronomie, Vermessungswesen, mathematische Geographie an den höheren Schulen* (III. Bd., Heft 4 der Abhandl.), actuellement sous presse.

DISCUSSION

M. Fujisawa, qui présidait la première partie de la séance, remercia M. Smith, au nom de l'assemblée, pour son exposé si documenté, puis il ouvrit la discussion sur cet intéressant rapport.

M. Laisant estime qu'il eût été préférable de faire porter l'enquête sur l'ensemble de l'enseignement, depuis la première initiation, tandis que la Sous-commission a cru devoir limiter l'étude à l'enseignement moyen. Puis MM. Thaer et Lietzmann développent et complètent certains passages du rapport de M. Smith pour ce qui concerne tout particulièrement l'Allemagne, M. E. Dintzl pour l'Autriche, M. Bioche pour la France, MM. Siddons et Carson pour l'Angleterre et M. Goldziher pour la Hongrie. Envisageant la question dans son ensemble, M. v. Dyck (Munich) tient à faire constater que les discussions soulevées par la Commission montrent le rôle utile qu'elle a déjà exercé jusqu'ici en appelant les mathématiciens du monde entier à réfléchir sur les questions si importantes des méthodes et des plans d'étude de l'enseignement mathémathique.

M. C.-A. Laisant (Paris). — Je ne voudrais à aucun prix entraver, ni même retarder, la discussion de l'intéressant et consciencieux rapport de notre collègue M. D.-E. Smith. Mais il est de mon devoir de vous dire les motifs pour lesquels il m'est impossible de prendre part à cette discussion, tout en reconnaissant qu'elle pourra produire quelques résultats utiles.

La question me semble mal posée, ou pour mieux dire posée d'une façon incomplète, ce qui, par cela même, la rend insoluble. Le rôle de l'intuition et de l'expérience dans l'éducation en général, dans l'enseignement mathématique notamment, est l'un des problèmes capitaux de la pédagogie. Mais restreindre l'examen de ce problème à la considération des écoles moyennes (ou secondaires) c'est le rendre insoluble, et méconnaître les conditions psychologiques et physiologiques du développement cérébral.

Suivant, en effet, qu'un enfant qui débute dans l'enseignement secondaire aura subi une préparation antérieure de telle ou telle nature, les procédés pédagogiques que vous pourrez lui appliquer utilement différeront du tout au tout. Si chez cet enfant, depuis le début, on a fait appel à l'intuition et à l'expérience, la continuation sera toute naturelle, et il n'y aura pour ainsi dire qu'à mettre l'élève en garde contre le danger des surprises, si le raisonnement logique n'exerce pas un contrôle assez sévère.

Si votre élève, au contraire, n'a acquis ses notions mathématiques antérieures que dogmatiquement et à force de mémoire — comme cela arrive trop souvent hélas — l'intuition et l'expérience exigeront de sa part de nouveaux efforts pour être mises en œuvre, en outre; il les considérera avec une sorte de dédain, comme des procédés manuels bons pour un apprentissage, mais indignes de lui. Et je n'hésite même pas à dire, moi partisan irréductible de l'intuition et de l'expérience introduites dans l'enseignement, que, dans certains cas particuliers, cette introduction peut devenir dangereuse.

La nature ne connaît pas nos divisions artificielles, nos classifications souvent arbitraires. Il est impossible de négliger le passé en s'occupant du présent, sous peine de grosses déceptions. Un architecte qui s'occuperait des matériaux à employer pour la construction des étages d'un édifice, sans se soucier de l'examen des fondations, serait d'une imprudence sans pareille.

Je crois donc que la Commission s'est trompée, en ne donnant pas à la question l'ampleur qu'elle comporte. Mais je suis sûr que le débat sera repris ultérieurement dans les conditions qu'entraîne fatalement la nécessité même des choses. Soit au prochain congrès, soit dans des réunions préparatoires qui auront lieu auparavant, la question se posera dans toute son unité, comme elle doit être posée. La discussion actuelle, je le répète, sera un élément utile, mais elle ne saurait avoir un caractère définitif.

Je regrette l'erreur commise par la Commission. Je serais cependant désolé qu'on pût attribuer à ma brève intervention une pensée d'obstruction qui est bien loin de moi.

Allemagne : *Remarques* de M. A. Thaer (Hambourg). — Mehrfach ist dem Schmerze über den Tod des Herrn P. Treutlein wohltuender Ausdruck verliehen worden, ganz besonders wird er aber heute bei der Besprechung von « Anschauung und Experiment im mathematischen Unterricht » vermisst werden, denn er ist seit mehr als einem Menschenalter der Führer dieser Bewegung in Baden gewesen und durch die Unterstützung des Herrn F. Klein für ganz Deutschland geworden. Es ist als ein Glück zu bezeichnen, dass es ihm noch vergönnt war sein Buch « Der geometrische Anschauungsunterricht als Unterstufe eines zweistufigen geometrischen Unterrichts an unseren höheren Schulen » zu vollenden. Das Werk ist zu sehr der Ausdruck der hinreissenden Persönlichkeit des Herrn Treutlein, als dass es einem andern möglich wäre eine einigermassen entsprechende Schilderung zu geben. Er empfiehlt darin die in Österreich-Ungarn seit einem halben Jahrhundert mit Erfolg erprobte Methode der Zweistufigkeit des geometrischen Unterrichts. Aber auch wer in diesem Punkte von ihm abweicht, wird dem Buche ausserordentlich viel entnehmen vor allem die Richtigkeit der feinen Charakterisierung, die Herr D. E. SMITH von der deutschen Art des Reform- Unterrichts gibt, « To mix the intuitional and the deductive work ». Herr F. KLEIN hat in dem Einführungswort zu dem Buche des Herrn P. TREUTLEIN jedes Missverständnis über den deutschen Anschauungsunterricht durch folgende Worte auszuschliessen versucht : « Der Kunst des Lehrers bleibt es überlassen, von den anschauungsmässigen Elementen schon während des vorbereitenden Unterrichts allmählich zur logischen Erfassung der Zusammenhänge überzuleiten, sodass nicht nur eine vorläufige Kenntnisnahme der grundlegenden räumlichen Vorstellungen, sondern unmittelbar eine tragfähige Grundlage für den höheren geometrischen Unterricht gewonnen ist, der, wie es Max Simon einmal ausdrückt, eine chemische Verbindung von Anschauung und Logik darstellen soll. »

Aber nicht bloss in diesem Punkt sondern auch in den übrigen Beziehungen gibt der Bericht des Herrn D. E. SMITH bei aller Kürze ein treues und klares Bild über den gegenwärtigen Stand der Reformbewegung in Deutschland, soweit er sich aus den umfassenden und eingehenden I. M. U. K. Berichten ersehen lässt und die Abschnitte « Anschauung und Experiment » betrifft.

Es könnte also genügend erscheinen, wenn ich hier nur noch einmal zur näheren Aufklärung auf die I. M. U. K. Abhandlungen hinweise, die neben den Einzelberichten die hier behandelten Fragen in mehr zusammenfassender Darstellung bringen, das sind die Arbeiten der Herren HOFFMANN, LIETZMANN, SCHIMMACK, ZÜHLKE.

So nahe es liegt, den Stand der Reform aus den Lehrbüchern zu beurteilen, sie können, wie auch schon die Berichte der Herren LIETZMANN, SCHIMMACK und ZÜHLKE hervorgehoben haben, nicht als unbedingt treues Bild gelten, selbst die ausdrücklich als Reformlehrbücher bezeichneten, so treffliche Wegweiser sie sind. Denn bei der Kürze der Zeit basieren sie zu sehr auf der subjektiven Ansicht der Verfasser und der im günstigsten Fall kurzfristigen Erfahrung eines ersten Experiments. Der lebendige Austausch von Erfahrungen beim Besuch von Schulen und auf Versammlungen ist jedenfalls als Ergänzung wünschenswert und tatsächlich von den oben genannten Herren benutzt worden ebenso wie die eingehenden Antworten in den versandten Fragebogen. Auch ich kann natürlich nur subjektive Eindrücke mitteilen, aber ich möchte betonen, dass sie nur zum kleinsten Teil dem eigenen Unterricht ihren Ursprung verdanken.

Der Ruf nach Anschauung hat zunächst zur Herstellung zahlreicher künstlicher Anschauungsmittel geführt. Mir scheint die herschende Tendenz jetzt dahin gehen zu wollen, die natürlichen Anschauungsmittel mehr zu benutzen, *den Schüler anzuleiten mathematische Begriffe aus seiner Umgebung zu abstrahieren und wieder auf sie anzuwenden* Wichtig ist, dass man sich nicht auf den Schulraum beschränkt, deshalb ist der *Feldmessunterricht* so fruchtbar, weil er ins Freie führt. Möglich ist er auch in der Grossstadt. Neben der allerdings nicht ganz zu vernachlässigenden Benutzung genauerer Instrumente — Messstange und Sextant verdienen vielleicht den Vorzug vor Bandmass und Theodolit — ist die einfache Zählung der Schritte, das aus dem Zeichenunterricht wohlbekannte Verfahren der Bestimmung scheinbarer Grössen mit Hilfe des ausgestreckten Armes und eines Masstabes zu pflegen. Benutzt man als Massstab den logarithmischen Rechenstab (sliding rule), so kann man die Winkel besonders bis 5° überraschend genau bestimmen.

Was die *Astronomie* betrifft, so möchte ich feststellen, dass ich unter den etwa 20 neu gebauten Schulen, die ich in Deutschland gesehen, auch höhere Mädchenschulen, keine ohne astronomisches Observatorium gefunden habe, zum Teil auch mit Präzisionsinstrumenten ausgestattet, was mir aber nicht das Wesentliche erscheint. Die Hauptsache bleibt ja, dass der durch das Copernikanische System hochmütig auf alle unmittelbare Anschauung herabblickende Schüler erst einmal wieder einigermassen im Ptolemäischen heimisch wird, die Drehung des Himmelsgewölbes, die Bewegung von Mond und Sonne, einiger Planeten, der Jupitermonde roh messend verfolgt. Ideal ist gewiss ein Unterricht, wie ihn Herr Hoffmann schildert, ich fürchte nur, nicht 5 % der Schulen werden ihn adoptieren, und mancher wird sich ganz abhalten lassen, weil er den Vorwurf der Unwissenschaftlichkeit fürchtet, wenn er es nicht so vollendet wie Herr Hoffmann macht. Das beste astronomische Schulobservatorium ist ein flaches Dach, zugleich eine vorzügl iche Basis für die Pothenotsche Aufgabe und, wenn ein sonstiger Turm zur Verfügung steht, für die Hansensche.

Für *Linearzeichnen und Darstellende Geometrie* möchte ich auf Herrn Zühlkes Arbeit verweisen und nur erwähnen, dass man neuerdings versucht, dieZentralperspektive nicht an den Schluss, sondern an den Anfang der Körperdarstellung zu stellen, älteren Anregungen des Württembergers Hertter und des Berliners Martus folgend. Für die Versuche spricht, dass dem Schüler aus dem Zeichenunterricht die Praxis geläufig ist und dass die Bilder anschaulicher und wohlgefälliger sind als die axonometrischen. Dagegen spricht, dass die Theorie schwieriger ist. Für die Darstellung der Kugel, besonders zur korrekten zeichnerischen Lösung astronomischer Aufgaben gewinnt die stereographische Projektion etwas mehr Freunde. Da vielfach Mathematiker sich jetzt in Deutschland auch die Lehrbefähigung in Geographie erwerben, wird Kartenprojektion und Kartenlesen auch im mathematischen Unterricht gepflegt.

Das *Zeichnen von Kurven* und graphischen Lösungen wird nach der Ansicht vieler bereits übertrieben. Vor allem sind statistische Kurven vielfach eher geeignet, den Begriff der Funktion zu verschleiern, als ihn zu enthüllen. In der Geographie sind sie gewiss nützlich, aber in der Mathematik eigentlich erst dann, wenn sie zur Feststellung einer funktionalen Abhängigkeit führen. Bei Kursschwankungen eines Industriepapiers dürfte das schwer sein, aber z. B. aus der Gold- und Silberproduktion auf den Preis des Silbers, soweit er nicht durch Börsenspekulationen alteriert ist, zu schliessen, ist für einen Schüler der Mittelklassen nicht zu schwer.

Es ist richtig, was Herr D. E. Smith sagt, dass der *logarithmische Rechenstab* in Deutschland bis vor Kurzem noch wenig verbreitet war, Schuld war erst der hohe Preis. Noch vor 10 Jahren kostete ein guter Stab 12 ½ M. Dann kamen billige und minderwertige Fabrikate, die erst recht den Rechenstab diskreditierten. Seit der Preis für gute Stäbe mit trigonometrischer Einrichtung auf 5 M. herabgegangen ist, nimmt ihr Gebrauch ausserordentlich zu. In Bayern ist er durch Herrn v. Dyck für alle Realanstalten obligatorisch gemacht. In einzelnen Schulen hat er aus den Oberklassen alle logarithmischen, trigonometrischen, Quadrat- und sonstigen Tafeln verdrängt. Für physikalische und Versicherungs-Aufgaben sind allerdings noch Tafeln nötig. Eine grosse Rolle spielt der Rechenstab auch gerade bei der graphischen Lösung numerischer Gleichungen, algebraischer sowohl wie transzendenter.

Wenn natürlich auch der Schüler die Kurven selbst zeichnen soll, möchte ich doch erwähnen, dass ein von Herrn J. Schröder in Hamburg konstruierter Apparat gute Dienste tun kann. Die Kurve wird aus Draht in einer ebenfalls aus Drähten gebildeten Ebene hergestellt. Macht man die Ebene um die Gerade $x - y = 0$ drehbar, so verwandelt sich jede Kurve durch Drehung um 180° im Raum in die inverse, also e^x in log nat x, sin x in arc sin x, tg x in arc tgx, u. s. w.

Quadratisches Papier wird sehr stark benutzt, Die Flächenbestimmung durch Abzählung findet z. B. Verwendung bei der Integralrechnung in der Auswertung bestimmter Integrale. Hier ist es nicht nur zur Prüfung eines erhaltenen Resultats nützlich, sondern es zeigt dem Schüler auch einen Ausweg, wo die ihm bekannten Formeln versagen.

Autriche : *Remarques de* M. E. Dintzl (Vienne). — Wenn ich zunächst dem zur Diskussion gestellten Programm die Frage herausgreife, wie der aus Unterricht in der *Darstellenden Geometrie* an den höheren Lehranstalten betrieben wird, so geschieht es deshalb, weil diese Disziplin in *Oesterreich* insbesondere an den Realschulen schon seit langem mit grosser Sorgfalt gepflegt wird und auch in der gegenwärtigen Form ganz charakteristische Merkmale aufweist. Ich könnte mich zwar damit begnügen, auf die ausführlichen Berichte von Prof. Müller und Adler in den Veröffentlichungen der österreichischen Subkommission hinzuweisen. Es gibt aber noch ein plastischeres Mittel, die Unterrichtsmethode in diesem Gegenstande vorzuführen, d. i. an der Hand von Zeichnungen, welche von den Schülern in den Unterrichtsstunden wirklich ausgeführt worden sind. Ich habe darum solche Schülerzeichnungen von vier Wiener Realschulen (2. Staatsrealschule im 7., 13. und 18. Bezirk) und einem Wiener Realgymnasium (2. Bezirk) mitgebracht und lade Sie ein, dieselben zu besichtigen. Was Ihnen bei der Durchsicht dieser Zeichnungen vor allem auffallen wird, das sind die vielen Darstellungen technischer Objekte, z. B. von Gerüstteilen, Säulen, Gesimsen, ja von ganzen Häusern und Kapellen. Dies ist ein besonderes Merkmal des modernen Unterrichtes und hängt aufs engste mit den Reformideen zusammen, welche Profes. Dr. Emil Müller von der technischen Hochschule in Wien inbezug auf den Unterricht in der Darstellenden Geometrie an den technischen Hochschulen vertritt. Solche Darstellungen einfacher technischer Objekte erwecken, wie von Seite der Lehrer stets versichert wird, das lebhafteste Interesse der Schüler und zwar aus einem doppelten Grunde. Einmal ist es klar, dass Wirklichkeitsaufgaben, hier also einfache Beispiele aus der Architektur, den Schüler mehr fesseln, als theoretische Aufgaben. Dazu kommt aber noch ein psychologisches Moment, d. i. die ständige Kontrolle, welche der Schüler aus sich selbst, aus seiner unmittelbaren Anschauung heraus auszuüben vermag. Das Bild eines technischen Objektes ist, wie einmal gesagt wurde, einem Porträt vergleichbar, an dem man bei jedem Striche, den man zeichnet, sieht, ob derselbe richtig ist oder nicht. Sie dürfen aber daraus nicht den Schluss ziehen, als ob der Unterricht in einer systemlosen Aneinanderreihung solcher Aufgaben bestünde. Dies widerlegen am besten die mitgebrachten Zeichnungen.

Ich wende mich nun den übrigen Fragen zu. Hiebei kann ich auf Details umso leichter verzichten, als ja Prof. D.-E. Smith in seinem ausgezeichneten Berichte dieselben für unser Land in klarer Weise beantwortet. Hier nur einige Bemerkungen allgemeiner Natur. Es ist whar, dass an den österreichischen Mittelschulen Anschauung und Experiment eine grosse Rolle spielen und zwar nicht bloss auf der dreijährigen Unterstufe, welche wir für eine ungemein wertvolle Institution halten, sondern auch in den mittleren und oberen Klassen, wo sie in der Arithmetik in einer vorwiegend graphischen Behandlung des Zahl- und Funktionsbegriffes, in der Geometrie in der Pflege des zeichnerischen Momentes zum Ausdrucke kommen. Die Bewegung schreitet in dieser Richtung gegenwärtig eher vor als zurück. Es ist augenblicklich schwer möglich, irgend eine Kritik auszuüben und auf die beiden Hauptfragen, welche Prof. D.-E. Smith in seinem Referate aufgeworfen hat, präzise Antworten zu geben, da ja die Reformbewegung in Oesterreich noch nicht so langen Datums ist. Aber eines kann heute schon gesagt werden. Wir stehen vor einer Schwierigkeit, um nicht zu sagen vor einer Gefahr, nämlich davor, dass durch das beständige Hereinziehen immer neuer Gegenstände der angewandten Mathematik — wie sie durch das praktische Leben gefordert werden — der systematische Unterricht mehr, als gut ist, eingeengt und bedrückt wird. Wie soll man nun dieser Gefahr wirksam entgegnen? In dieser Hinsicht hat vor kurzem Professor Sobotka von der böhmischen Universität in Prag Ideen entwickelt, welche sehr beachtenswert sind und für die künftige Entwicklung in Oesterreich von Bedeutung sein können. Sobotha spricht zunächst von der Bedeutung des Experimentes für den eigentlichen mathematischen Unterricht in Arithmetik und Geometrie und sagt, dass « das Experiment (Demonstration von Modellen, graphische Darstellungen, etc.) von Lehrer und Schüler zu pflegen sei, aber nur soweit es das selbständige Erfassen und die Ausbildung des Raumanschauungsvermögens unterstützt, zum Nachdenken zwingt oder dem Gegenstande der Betrachtung neue Seiten abzugewinnen geeignet ist. somit als *Behelf* und nicht als *ausschliessliches Mittel zum Zweck* ». « Was die *Anwendung auf technische Probleme* oder solche, die dem praktischen Leben überhaupt entnommen sind, anlangt, so sei vorderhand — ähnlich wie es in der Physik teilweise der Fall ist, ein besonderes Praktikum der angewandten Mathematik in jeder Mittelschule einzuführen. Hier käme das Experiment in ausgedehntem Masse und in systematischer Anwendung zur Geltung. In weiterer Folge sei jedoch anzustreben, dass angewandte Mathematik mit praktischen Uebungen als Lehrgegenstand, wenn nicht allgemein, so doch wenigstens an den Realschulen, *obligatorisch* eingeführt werde. »

Das sind in aller Kürze die Ideen von Sobotka, soweit sie mit den zur Diskussion gestellten Fragen in Zusammenhang stehen. Man mag gegen diese Vorschläge vielleicht vom Standpunkt der Fusion Bedenken erheben. Dieselben scheinen aber nicht schwerwiegend zu sein — umsoweniger als wir ja in dem Unterrichte in der Darstellenden Geometrie und in seinem Verhältnis zum systematischen Unterricht in der Stereometrie ein gutes Vorbild besitzen. Sicher haben diese Ideen das eine für sich, dass auf diese Weise der Intuition und dem Experimente eine genau fixierte Stellung zugewiesen ist, eine Stellung, welche auch der so wichtigen Erziehung der jungen

Leute im logischen, deduktiven Denken genügend Raum zur Entwicklung lässt.

France. — M. Ch. Bioche (Paris) insiste sur l'importance qu'on donne en *France* à la Géométrie descriptive et au Dessin géométrique.

L'exécution d'un dessin géométrique, ou d'une épure oblige l'élève

1° à constater la nécessité d'un raisonnement logique pour établir les faits géométriques, par exemple l'existence d'un cercle passant par 3 points;

2° à se préoccuper de déduire de la solution théorique d'un problème une construction qui soit à la fois simple, précise et susceptible d'être effectuée dans les limites de la feuille de papier.

Les professeurs de mathématiques ont fréquemment demandé que l'enseignement du dessin géométrique soit confié aux professeurs de mathématiques pour que cet enseignement facilite et complète l'enseignement théorique.

Iles britanniques. — M. G.-S.-L. Carson (Tonbridge) said that, in in *England* at any rate, there appeared to be confusion between the terms experiment and intuition. Results such as the angle properties of parallel lines, which are at once accepted by a child of 12 if expressed in non-technical terms, are made the subject of numerical measurement with a protractor and yet referred to as intuitive, instead of experimental. He was strongly of opinion that only intuitions in the proper sense of the term should be taken as the basis of a first course of formal Geometry, and that all possible intuitions should be taken as the basis. In this way only could the appearance of proving the obvious, so dangerous and destructive to the logical sense of a young child, be avoided.

He demurred most strongly to the suggestion that logic might be less rigorous at an early age. More postulates than the minimum necessary might and should be accepted, but the processes of deduction based on these postulates should be entirely rigorous from the outset. Any want of rigour in deduction is detected sooner or later, and with bad results; whereas analysis of the interconnection of the set of postulates which have been assumed is, at a later stage, a natural and interesting course for the pupil.