Sato-Tate in the higher dimensional case : elaboration of 9.5.4 in Serre's Nx(p) book

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SATO-TATE IN THE HIGHER DIMENSIONAL CASE: ELABORATION OF 9.5.4 IN SERRE'S $N_X(p)$ BOOK

by Nicholas M. KATZ

1. INTRODUCTION

In the very last paragraph of Serre's book *Lectures on* $N_X(p)$, he writes "An interesting fact is that the Sato-Tate conjecture is sometimes easier to prove in the higher dimensional case ($d > 1$) than in the number field case, thanks to the information given by the geometric monodromy (as done by Deligne in characteristic p, cf. [De 80])." The purpose of this note is to spell out how this is done. In the higher dimensional case, one can bring to bear monodromy techniques. It turns out that a mild hypothesis "(H)" on the geometric monodromy is all that is needed; one gets a natural "Sato-Tate group" K in the sense of [Se-$N_X(p)$, 8.2.2], in whose space of conjugacy classes the equidistribution takes place. Questions of modularity do not arise.

The prototypical situation to be dealt with was first considered by Birch [B], where he looked at the universal family of elliptic curves in Weierstrass form $y^2 = x^3 - ax - b$, a, b indeterminates, over the ground ring $W := \mathbf{Z}[a, b, 1/6, 1/(4a^3 - 27b^2)]$. It is the Spec of this ground ring which is "higher dimensional", and our concern is with equidistribution properties of the unitarized Frobenius conjugacy classes attached to the closed points $\mathfrak{P}$ of Spec(W). In this example, we may view the parameter space $X :=$ Spec(W) as a "scheme over S" in various ways, for example, as a scheme over $\mathbf{Z}[1/6]$ or as a scheme over $\mathbf{Z}[1/6, a]$ or as a scheme over $\mathbf{Z}[1/6, b]$. The basic object of study in this example is the *lisse sheaf* $\mathcal{F}$ on the parameter space $X :=$ Spec(W) which is the "H^1 along the fibres" of the Weierstrass family. Birch showed that as p grows, the $\mathbf{F}_p$ points of $X :=$ Spec(W) have unitarized Frobenii which are closer and closer to being distributed according to Sato-Tate.

In the general story to be developed below, Birch's result is of the type we call "packet by packet", when viewing $X := \mathrm{Spec}(W)$ as a scheme over $\mathbf{Z}[1/6]$.

In these higher dimensional equidistribution questions, there are three sorts of equidistribution which are relevant, which we call "packet by packet", "packetwise", and "classical". The first implies the second, cf. Prop. 5.2. In generic characteristic zero, the second is equivalent to the third, cf. Lemma 3.6. The third is false in equicharacteristic $p > 0$, cf. Remark 3.7.

2. The general setting: review of Pink's theorem

In this section, we consider the following general situation. We are given a noetherian normal connected scheme S, and a smooth S-scheme $f\colon X \to S$ with geometrically connected fibres of dimension $d \geq 1$. We denote by η the generic point of S (i.e., η is the Spec of the function field $\kappa(S)$ of S), by $\overline{\eta}$ the Spec of a separable closure $\kappa(S)^{sep}$ of $\kappa(S)$, and by X_η and $X_{\overline{\eta}}$ the corresponding generic and geometric generic fibres of X/S. The scheme X is itself normal and connected, and we denote by ξ and $\overline{\xi}$ its generic and geometric generic points. Then $\overline{\xi}$ is also a geometric point of $X_{\overline{\eta}}$ and of X_η. We have morphisms of pointed (by $\overline{\xi}$) schemes

$$X_{\overline{\eta}} \to X_\eta \to X\,.$$

The fundamental groups are related as follows.

$$\pi_1(X_{\overline{\eta}}, \overline{\xi}) \lhd \pi_1(X_\eta, \overline{\xi})\,,$$

indeed we have the short exact sequence

$$1 \to \pi_1(X_{\overline{\eta}}, \overline{\xi}) \to \pi_1(X_\eta, \overline{\xi}) \to \mathrm{Gal}(\overline{\eta}/\eta) \to 1\,.$$

And we have

$$\pi_1(X_\eta, \overline{\xi}) \twoheadrightarrow \pi_1(X, \overline{\xi})\,,$$

because both these groups are quotients of the absolute Galois group of $\kappa(X)$. We also have a right exact sequence

$$\pi_1(X_{\overline{\eta}}, \overline{\xi}) \to \pi_1(X, \overline{\xi}) \to \pi_1(S, \overline{\eta}) \to 1\,,$$

cf. [Ka-La, Lemma 2]. Thus the image of $\pi_1(X_{\overline{\eta}}, \overline{\xi})$ in $\pi_1(X, \overline{\xi})$ is a normal subgroup of $\pi_1(X, \overline{\xi})$. We will denote this image group $\pi_1^{geom}(X/S, \overline{\xi}) \lhd \pi_1(X, \overline{\xi})$, hence we have a short exact sequence

$$1 \to \pi_1^{geom}(X/S, \overline{\xi}) \to \pi_1(X, \overline{\xi}) \to \pi_1(S, \overline{\eta}) \to 1\,.$$

If we take an arbitrary geometric point x of X, and a "chemin" from $\overline{\xi}$ to x, we get an isomorphism

$$\pi_1(X, \overline{\xi}) \cong \pi_1(X, x).$$

If we change the chemin, we change the isomorphism by an inner automorphism of either source or target. Because $\pi_1^{geom}(X/S, \overline{\xi}) \lhd \pi_1(X, \overline{\xi})$, the image in $\pi_1(X,x)$ of the normal subgroup $\pi_1^{geom}(X/S, \overline{\xi}) \lhd \pi_1(X, \overline{\xi})$ is a well-defined (i.e. independent of the choice of chemin) normal subgroup of $\pi_1(X,x)$, which we denote $\pi_1^{geom}(X/S,x) \lhd \pi_1(X,x)$. Because this subgroup is normal, to say that a given subgroup Γ of $\pi_1(X,x)$ lies in $\pi_1^{geom}(X/S,x)$ is a meaningful statement (i.e., independent of the choice of chemin from x to $\overline{\xi}$). More concretely, a given subgroup Γ of $\pi_1(X,x)$ lies in $\pi_1^{geom}(X/S,x)$ if and only if every $\pi_1(X,x)$-conjugate of Γ lies in $\pi_1^{geom}(X/S,x)$, if and only if some $\pi_1(X,x)$-conjugate of Γ lies in $\pi_1^{geom}(X/S,x)$.

The following theorem of Pink is proven in [Ka-ESDE, 8.18.2], despite being imprecisely stated there. For the reader's convenience, we give the proof.

THEOREM 2.1. *Let ℓ be a prime number (not assumed invertible on S), $\mathcal{F}$ a lisse $\overline{\mathbf{Q}_\ell}$ sheaf on X of rank $n \geq 1$.*

(1) *For every geometric point s of S, and every geometric point x of the fibre X_s, when we view $\mathcal{F}$ as a representation $\rho_x \colon \pi_1(X,x) \to \mathrm{Aut}_{\overline{\mathbf{Q}_\ell}}(\mathcal{F}_x)$, we have an inclusion*

$$\rho_x(\pi_1(X_s,x)) \subset \rho_x(\pi_1^{geom}(X/S,x))$$

inside $\rho_x(\pi_1(X,x))$.

(2) *There exists a dense open set $U \subset S$ such that if the geometric point s lies in U, then for every geometric point x of the fibre X_s we have an equality*

$$\rho_x(\pi_1(X_s,x)) = \rho_x(\pi_1^{geom}(X/S,x))$$

inside $\rho_x(\pi_1(X,x))$.

Proof. We first prove (1). The lisse $\overline{\mathbf{Q}_\ell}$ sheaf $\mathcal{F}$ descends to a lisse E_λ-sheaf, for some finite extension $E_\lambda/\mathbf{Q}_\ell$, then to a lisse $\mathcal{O}_\lambda$-sheaf $\mathcal{F}_0$ for $\mathcal{O}_\lambda$ the ring of integers in E_λ. So it suffices to prove (1) for each of the lisse sheaves $\mathcal{F}_0/\ell^n\mathcal{F}_0$. Fixing one such, we have a finite group G, and a surjective homomorphism $\rho_{x,n} \colon \pi_1(X,x) \twoheadrightarrow G$. Denote by $H \lhd G$ the image of $\pi_1^{geom}(X/S,x)$. Then the quotient homomorphism from $\pi_1(X,x)$ onto G/H factors through $\pi_1(S)$. Replacing S by the finite étale covering S_1 of itself which trivializes this homomorphism, replacing s by a geometric point of S_1 lying over s, replacing X/S by X_{S_1}/S_1, and replacing x by a geometric point x_1 of X_{S_1} lying over s_1, we reduce to treating the case when $\pi_1^{geom}(X/S,x)$ and $\pi_1(X,x)$ have the **same** image, here H. In this case, the asserted inclusion of (1) is just the inclusion of $\rho_x(\pi_1(X_s,x))$ in $\rho_x(\pi_1(X,x))$.

We next show that for our fixed $\mathcal{F}_0/\ell^n\mathcal{F}_0$, there is an open dense set $U_n \subset S$ over which $\rho_{x,n}(\pi_1(X_s,x)) = \rho_{x,n}(\pi_1^{geom}(X/S,x)$. As above, we may first base change to S_1, then take for $U_n \subset S$ the image of the dense open $U_{n,1} \subset S_1$ we find. Then we consider the finite étale H-torsor $E \to X$. The equality $\rho_{x,n}(\pi_1(X_s,x)) = \rho_{x,n}(\pi_1^{geom}(X/S,x))$ holds precisely at the geometric points s over which the fibre E_s (of E viewed as S-scheme, say $h\colon E \to S$) is connected, or, equivalently, irreducible (remember E/S is lisse, everywhere of relative dimension d). The number of irreducible components of E_s is the dimension of the stalk at s of the constructible sheaf $R^{2d}h_!\overline{\mathbf{Q}_\ell}$. By construction, this dimension is one at $\overline{\eta}$, so is one on some dense open set. [See also [De-Weil II, 1.11.5] for another approach to this "finite" case.]

To end the proof, we use Pink's Lemma [Ka-ESDE, 8.18.3], which insures that we can take for U the open set U_n for n sufficiently large. [The statement is that given a closed subgroup $K \subset GL(n, \mathcal{O}_\lambda)$, (here the image $\rho_x(\pi_1^{geom}(X/S,x))$, there is an integer ν with the following property: if a closed subgroup H (here $\rho_x(\pi_1(X_s,x))$) of K maps onto the image of K in $GL(n, \mathcal{O}_\lambda/\ell^\nu\mathcal{O}_\lambda)$, then $H = K$. See also [T, proof of Thm. 2], [Se-CP4, &133, pp. 1–2], and [Se-MW, 10.6] for arguments of this type.] □

3. Formulation of two equidistribution theorems

In this section, we suppose further that the noetherian normal connected scheme S is a $\mathbf{Z}[1/\ell]$-scheme of finite type. We fix a field embedding $\iota\colon \overline{\mathbf{Q}_\ell} \subset \mathbf{C}$, and a real number w. We fix a lisse $\overline{\mathbf{Q}_\ell}$ sheaf $\mathcal{F}$ on X of rank $n \geq 1$ which is ι-pure of weight w. For a fixed geometric point x of X, we denote by

$$G_{arith,X,x} \subset \mathrm{Aut}_{\overline{\mathbf{Q}_\ell}}(\mathcal{F}_x)$$

the Zariski closure in $\mathrm{Aut}_{\overline{\mathbf{Q}_\ell}}(\mathcal{F}_x)$ of the image $\rho_x(\pi_1(X,x))$, and by

$$G_{geom,X/S,x} \lhd G_{arith,X,x}$$

the Zariski closure in $G_{arith,X,x}$ of the image $\rho_x(\pi_1^{geom}(X/S,x))$.

For each closed point $\mathfrak{p}$ of S, with residue field $\mathbf{F}_\mathfrak{p}$ and geometric point $\overline{\mathfrak{p}} = \mathrm{Spec}(\overline{\mathbf{F}_\mathfrak{p}})$ lying over it, we have the fibre $X_\mathfrak{p}/\mathbf{F}_\mathfrak{p}$ and the geometric fibre $X_{\overline{\mathfrak{p}}}/\overline{\mathbf{F}_\mathfrak{p}}$. For a geometric point x of $X_{\overline{\mathfrak{p}}}$, we denote by

$$G_{geom,X_\mathfrak{p},x} \lhd G_{arith,X_\mathfrak{p},x}$$

the Zariski closures in $G_{arith,X,x}$ of the images of $\pi_1(X_{\overline{\mathfrak{p}}},x)$ and $\pi_1(X_\mathfrak{p},x)$ respectively.

In what follows, we will use some choice of chemin from x to the geometric generic point $\overline{\xi}$ of X, drop the x from the notation, and view all of the groups $G_{geom,X_\mathfrak{p}}$, $G_{arith,X_\mathfrak{p}}$, $G_{geom,X/S}$, $G_{arith,X}$ as Zariski closed subgroups of $G_{arith,X}$, with inclusions

$$G_{geom,X_\mathfrak{p}} \lhd G_{arith,X_\mathfrak{p}} \subset G_{arith,X}\ ,$$

$$G_{geom,X_\mathfrak{p}} \subset G_{geom,X/S}\ ,$$

$$G_{geom,X/S} \lhd G_{arith,X}\ ,$$

all well defined up to $G_{arith,X}$-conjugation.

By Pink's theorem we not only have

$$G_{geom,X_\mathfrak{p}} \subset G_{geom,X/S}$$

for every closed point $\mathfrak{p}$ of S, but we have equality for all those $\mathfrak{p}$ lying in some dense open $U \subset S$.

Recall (cf. [De-Weil II, 1.3.9 the first paragraph of its proof, and 3.4.1 (iii)]) that for each closed point $\mathfrak{p}$ of S, the group $G_{geom,X_\mathfrak{p}}$ is semisimple. In view of Pink's theorem, the group $G_{geom,X/S}$ is semisimple.

For the rest of this note, we impose the following hypothesis[1]) (H) on our data $(X/S, \mathcal{F}, \iota)$:

$$\text{Hypothesis } (H) : \rho(\pi_1(X)) \subset \mathbf{G}_m G_{geom,X/S}\ ,$$

or, equivalently,

$$G_{arith,X} \subset \mathbf{G}_m G_{geom,X/S}\ .$$

For each closed point $\mathfrak{p}$ of S, we thus have

$$\rho(\pi_1(X_\mathfrak{p})) \subset \mathbf{G}_m G_{geom,X/S}\ .$$

[1]) As we will see, the role of Hypothesis (H) is to allow us to make use of known equidistribution theorems on the fibres of X/S. See section 9 for examples where Hypothesis (H) does not hold (and where looking along the fibres does not help).

LEMMA 3.1. *For each closed point* $\mathfrak{p}$ *of* S, *there exists* $\alpha_\mathfrak{p} \in \overline{\mathbf{Q}_\ell}^\times$ *with the following properties:*

(1) *For each finite extension* $k/\mathbf{F}_\mathfrak{p}$, *and each point* $t \in X_\mathfrak{p}(k)$,

$$\rho(Frob_{k,t})/\alpha_\mathfrak{p}^{deg(k/\mathbf{F}_\mathfrak{p})} \in G_{geom,X/S}.$$

(2) *For each closed point* $\mathfrak{P}$ *of* $X_\mathfrak{p}$ *(i.e., for each closed point* $\mathfrak{P}$ *of* X *which lies above* $\mathfrak{p}$*),*

$$\rho(Frob_\mathfrak{P})/\alpha_\mathfrak{p}^{deg(\mathbf{F}_\mathfrak{P}/\mathbf{F}_\mathfrak{p})} \in G_{geom,X/S}.$$

Any $\alpha_\mathfrak{p}$ *satisfying either* (1) *or* (2) *is an* ℓ*-adic unit, i.e.* $\alpha_\mathfrak{p} \in \mathcal{O}^\times_{\overline{\mathbf{Q}_\ell}}$, *the lisse sheaf on* $X_\mathfrak{p}$ *given by*

$$\mathcal{G}_\mathfrak{p} := (\mathcal{F}|X_\mathfrak{p}) \otimes \alpha_\mathfrak{p}^{-deg}$$

is ι*-pure of weight zero, and its associated representation* $\rho_{\mathcal{G}_\mathfrak{p}}$ *maps* $\pi_1(X_\mathfrak{p})$ *to* $G_{geom,X/S}$.

Proof. If $X_\mathfrak{p}$ has an $\mathbf{F}_\mathfrak{p}$-rational point t, then any Frobenius element $Frob_{\mathfrak{p},t}$ is an element of degree one in $\pi_1(X_\mathfrak{p})$. Choose any $\alpha_\mathfrak{p} \in \overline{\mathbf{Q}_\ell}^\times$ such that $\rho(Frob_{\mathfrak{p},t})/\alpha_\mathfrak{p}$ lies in $G_{geom,X/S}$, this being possible by hypothesis (H). If not, use the fact that in all large enough extensions of $\mathbf{F}_\mathfrak{p}$, there are rational points. Denote by k_n the extension of degree n of $\mathbf{F}_\mathfrak{p}$. Then for n large, choose a point $t_n \in X_\mathfrak{p}(k_n)$ and a point $t_{n+1} \in X_\mathfrak{p}(k_{n+1})$. Then choose Frobenius elements $Frob_{k_{n+1},t_{n+1}}$ and $Frob_{k_n,t_n}$ in $\pi_1(X_\mathfrak{p})$. Then the element $\gamma_\mathfrak{p} := Frob_{k_{n+1},t_{n+1}} Frob^{-1}_{k_n,t_n}$ is an element of degree one in $\pi_1(X_\mathfrak{p})$. Choose any $\alpha_\mathfrak{p} \in \overline{\mathbf{Q}_\ell}^\times$ such that $\rho(\gamma_\mathfrak{p})/\alpha_\mathfrak{p}$ lies in $G_{geom,X/S}$, this being possible by hypothesis (H). Now (1) holds because for any Frobenius element $Frob_{k,t}$, the "ratio" $Frob_{k,t}\gamma_\mathfrak{p}^{-deg(k/\mathbf{F}_\mathfrak{p})}$ is an element of degree zero in $\pi_1(X_\mathfrak{p})$, so lies in $\pi_1^{geom}(X_\mathfrak{p})$, which in turn maps by ρ to $G_{geom,X/S}$ by Pink's theorem. Unscrewing all this, we get (1). We get (2) by repeating this argument with $Frob_{k,t}$ replaced by $Frob_\mathfrak{P}$.

To see that $\alpha_\mathfrak{p} \in \mathcal{O}^\times_{\overline{\mathbf{Q}_\ell}}$, we argue as follows. We know that $\mathcal{F}$ has an $\mathcal{O}_{\overline{\mathbf{Q}_\ell}}$-form, so $\det(\rho(\gamma_\mathfrak{p}))$ lies in $\mathcal{O}^\times_{\overline{\mathbf{Q}_\ell}}$. But $G_{geom,X/S}$ is semisimple, so det on it has finite order. Thus $\det(\rho(\gamma_\mathfrak{p})/\alpha_\mathfrak{p})$ is a root of unity, and hence $\alpha_\mathfrak{p}$ is an ℓ-adic unit. To see that $(\mathcal{F}|X_\mathfrak{p}) \otimes \alpha_\mathfrak{p}^{-deg}$ is ι-pure of weight zero, use the fact that it is ι-pure of some weight (namely $w - 2\log_q(|\iota(\alpha_\mathfrak{p})|)$ for $q = \#(\mathbf{F}_\mathfrak{p})$). So we can read its weight from the weight of its determinant. But $\rho_{\mathcal{G}_{\mathfrak{p}}}$ maps $\pi_1(X_{\mathfrak{p}})$ to $G_{geom,X/S}$, where every element has a determinant which is a root of unity. □

REMARK 3.2. There may be several choices of $\alpha_{\mathfrak{p}}$. Indeed, the indeterminacy is precisely the finite group $\mathbf{G}_m \cap G_{geom,X/S}$ (finite because $G_{geom,X/S}$ is semisimple). In what follows, we fix a choice of an $\alpha_{\mathfrak{p}}$ for each closed point $\mathfrak{p}$ of S.

We now use the embedding ι: $\overline{\mathbf{Q}_\ell} \subset \mathbf{C}$ to view $\mathbf{C}$ as a $\overline{\mathbf{Q}_\ell}$-algebra, so we can form the group $G_{geom,X/S}(\mathbf{C})$, which we view as a complex semisimple Lie group in the "classical" topology. We denote by K a choice of maximal compact subgroup of $G_{geom,X/S}(\mathbf{C})$.

LEMMA 3.3. *Let $\mathfrak{p}$ be a closed point of S, $\mathfrak{P}$ a closed point of X lying over $\mathfrak{p}$. Then the semisimplification (in the sense of Jordan decomposition) of $\iota(\rho(Frob_{\mathfrak{P}})/\alpha_{\mathfrak{p}}^{deg(\mathbf{F}_{\mathfrak{P}}/\mathbf{F}_{\mathfrak{p}})})$ in $G_{geom,X/S}(\mathbf{C})$ is conjugate in $G_{geom,X/S}(\mathbf{C})$ to an element of K, which is itself unique up to K-conjugacy.*

Proof. Because $(\mathcal{F}|X_{\mathfrak{p}})\otimes\alpha_{\mathfrak{p}}^{-deg}$ is ι-pure of weight zero, this semisimple element in the semisimple group $G_{geom,X/S}(\mathbf{C})$ has all its eigenvalues on the unit circle, so lies in a compact subgroup of $G_{geom,X/S}(\mathbf{C})$, hence lies in a maximal compact subgroup of $G_{geom,X/S}(\mathbf{C})$, and all such are $G_{geom,X/S}(\mathbf{C})$-conjugate. That all ways to conjugate this element into K lead to K-conjugate elements results from the Peter-Weyl theorem (that conjugacy classes in K are determined by their traces in all irreducible representations of K), Weyl's unitarian trick (that irreducible representations of K are the restrictions to K of (the image under ι of) irreducible representations of $G_{geom,X/S}$), and the fact that in any representation of $G_{geom,X/S}$, an element and its semisimplification have the same trace. □

Thus for each closed point $\mathfrak{P}$ of X, we obtain a K-conjugacy class $\theta_{\mathfrak{P}}$, whose definition involves the rescaling by the chosen $\alpha_{\mathfrak{p}}$, for $\mathfrak{p}$ the closed point of S lying under $\mathfrak{P}$.

We now formulate two theorems concerning the equidistribution properties of the conjugacy classes $\theta_{\mathfrak{P}}$.

To formulate the first, recall that for a scheme W of finite type over $\mathbf{Z}$, we denote by $|W|$ the set of its closed points, and by π_W: $\mathbf{R}_{>0} \to \mathbf{Z}$ the counting function

$$\pi_W(t) := \#\{\mathfrak{P} \in |W|,\ \mathbf{N}\mathfrak{P} \le t\}.$$

THEOREM 3.4 (Packetwise Equidistribution). *Suppose hypothesis (H) holds. For $t \in \mathbf{R}_{>0}$ large enough that $\pi_X(t) > 0$, denote by $\mu(\leq t)$ the probability measure on $K^{\#}$ defined by*

$$\mu(\leq t) := (1/\pi_X(t)) \sum_{\mathfrak{P} \in |X|, \mathbf{N}\mathfrak{P} \leq t} \delta_{\theta_{\mathfrak{P}}},$$

i.e., it is the measure "average a function over **all** *the closed points of norm at most t". Then as $t \to \infty$, the measures $\mu(\leq t)$ converge weakly to the induced "Haar measure" of total mass one on $K^{\#}$: for every continuous central $\mathbf{C}$-valued function f on K, we have the integration formula*

$$\int_K f\, d\mu_{Haar} = \lim_{t \to \infty} \int_K f\, d\mu(\leq t).$$

We call this first theorem "packetwise equidistribution" because in our successive approximating measures, we add on entire packets of closed points, namely all those of given norm, as we pass from one approximant to the next. The second theorem is "classical" equidistribution, but is valid only when we are in generic characteristic zero.

THEOREM 3.5 (Classical Equidistribution). *Suppose that X has generic characteristic zero and hypothesis (H) holds. As $\mathfrak{P}$ varies over the closed points of X, ordered by increasing $\mathbf{N}\mathfrak{P}$ (ties to be broken arbitrarily), the sequence of conjugacy classes $\theta_{\mathfrak{P}}$ is equidistributed in the space $K^{\#}$ of conjugacy classes of K for the induced "Haar measure" of total mass one.*

As Serre explains in [Se-$N_X(p)$, 9.2.1 small print], we have the following lemma.

LEMMA 3.6. *Suppose X has generic characteristic zero. Then Theorems* 3.4 *and* 3.5 *are equivalent.*

Proof. That Theorem 3.5 implies Theorem 3.4 is obvious. To show that Theorem 3.4 implies Theorem 3.5, we argue as follows. The scheme X is irreducible of some dimension $D = d + \dim(S) \geq 1$. As X has generic characteristic zero, Serre tells us [Se-$N_X(p)$, Cor. 9.2] that $\pi_X(t) \sim t^D/D\log(t)$. From this asymptotic, it follows that

$$\pi_X(t+1) - \pi_X(t) = o(\pi_X(t)).$$

Serre gives an algebro-geometric argument for this estimate, in the sharper form

$$\pi_X(t+1) - \pi_X(t) = O(t^{D-1}).$$

Denote by $g\colon X \to \mathrm{Spec}(\mathbf{Z}[1/\ell])$ the structural morphism. The fibres all have dimension $\leq D-1$, so by the constructibility of the higher direct images $R^i g_! \overline{\mathbf{Q}_\ell}$, their vanishing for $i > 2(D-1)$, the Lefschetz Trace Formula, and Deligne's fundamental estimates, we conclude that there exists an upper bound $M \in \mathbf{Z}$ for the sum of the compact Betti numbers of the fibres, and that we have the inequality $\#X(\mathbf{F}_{p^n}) \leq Mp^{n(D-1)}$ for all primes $p \neq \ell$ and all $n \geq 1$. Now $\pi_X(t+1) - \pi_X(t)$ can only be nonzero if the unique integer in $(t, t+1]$ is a prime power p^n, in which case we are counting closed points of norm p^n in X, and the number of these is trivially bounded by $\#X(\mathbf{F}_{p^n})$, which is $O(p^{n(D-1)}) = O(t^{D-1})$.

To deduce classical equidistribution from packetwise equidistribution, fix a continuous central function f on K which (by subtracting a constant and rescaling we may assume) has $\int_K f\, d\mu_{Haar} = 0$ and has $Sup_K|f(k)| \leq 1$. Fix $\epsilon > 0$. For t large enough, we have both

$$|\int_K f\, d\mu(\leq t)| \leq \epsilon \quad \text{and} \quad \pi_X(t+1) - \pi_X(t) \leq \epsilon\pi_X(t).$$

Then for any subset A of the set of closed points with norm in the interval $(t, t+1]$, we have

$$|\sum_{\mathbf{N}\mathfrak{P} \leq t} f(\theta_{\mathfrak{P}}) + \sum_{\mathfrak{P} \in A} f(\theta_{\mathfrak{P}})| \leq \epsilon\pi_X(t) + \#A$$

$$\leq \epsilon\pi_X(t) + (\pi_X(t+1) - \pi_X(t)) \leq 2\epsilon\pi_X(t) \leq 2\epsilon(\pi_X(t) + \#A). \qquad \square$$

REMARK 3.7. Here is a simple example, along lines suggested by Serre, to show that in equicharacteristic $p > 0$, classical equidistribution is false. Take an odd prime p, $S = \mathrm{Spec}(\mathbf{F}_p)$ and $X = \mathbf{G}_m/\mathbf{F}_p = \mathrm{Spec}(\mathbf{F}_p[x, 1/x])$. Pick a prime $\ell \neq p$ and view the quadratic character $\chi_2\colon \mathbf{F}_p^\times \to \pm 1$ as taking values in $\overline{\mathbf{Q}_\ell}^\times$, so giving a Kummer sheaf $\mathcal{L}_{\chi_2}$ on X. Here $G_{geom,X/S} = G_{arith,X} = \pm 1$. The class $\theta_{\mathfrak{P}}$ attached to a closed point $\mathfrak{P}$ of norm p^n is the following: such a closed point is an irreducible monic polynomial $f(x) \in \mathbf{F}_p[x]$ with $f(0) \neq 0$, and its $\theta_{\mathfrak{P}} \in \pm 1$ is $\chi_2((-1)^n f(0))$. According to the packetwise theorem, as we look at all closed points of norm at most p^n for large n, about half give $+1$ and about half give -1. On the other hand, as we will see in Theorem 5.1, it is also true that when we look at the closed points of norm precisely p^{n+1}, about half give $+1$ and about half give -1. But for any $p \geq 11$, there are *more*[2]) closed points of norm p^{n+1} than there are of all lower norms combined. So if we tag onto all the closed points of norm at most p^n only the half of the closed points of norm p^{n+1} giving $+1$, then of all these points, something like at least $2/3$ give $+1$ instead of -1.

4. Proof of the packetwise equidistribution theorem: first reduction

LEMMA 4.1. *For Z any proper closed subscheme of X, we have the estimate $\pi_Z(t) = o(\pi_X(t))$.*

Proof. The scheme X, being smooth with geometrically connected fibres over the connected normal $\mathbf{Z}[1/\ell]$-scheme of finite type S, is itself a connected, normal $\mathbf{Z}[1/\ell]$-scheme of finite type, so is irreducible of dimension $D = d + \dim(S)$. Any proper closed subscheme Z of X is a finite union of irreducible $\mathbf{Z}[1/\ell]$-schemes Z_i of finite type, each of dimension $d_i \leq D - 1$.

When S, or equivalently X, has generic characteristic zero, Serre tells us [Se-$N_X(p)$, Cor. 9.2] that $\pi_X(t) \sim t^D/D\log(t)$. For each Z_i which is itself of generic characteristic zero, we have $\pi_{Z_i}(t) \sim t^{d_i}/d_i \log(t)$. For an irreducible component Z_i which is an irreducible $\mathbf{F}_p$-scheme for some prime p, of dimension $d_i \leq D-1$, we argue as follows. If $d_i = 0$, then $\pi_{Z_i}(t)$ is bounded. If $1 \leq d_i < D-1$, then by Noether normalization applied to affine pieces we have an estimate of the form $\#Z_i(\mathbf{F}_{p^n}) = O(p^{nd_i})$ for variable n. So trivially the number of closed points of norm at most p^n is $O(\sum_{j=1}^{n} p^{jd_i})$. The inner sum $\sum_{j=1}^{n} p^{jd_i} = (p^{d_i}/(p^{d_i}-1))(p^{nd_i}-1) \leq 2p^{nd_i}$. Thus $\pi_{Z_i}(t) = O(t^{d_i})$ whenever t is a power of p, and hence for all $t > 0$ (because $\pi_{Z_i}(t)$ is increasing, and only changes value when t is a power of p).

When S, or equivalently X, has generic characteristic $p > 0$, then the algebraic closure of $\mathbf{F}_p$ in the function field $\kappa(X)$ is a finite field $\mathbf{F}_q$, and $X/\mathbf{F}_q$ is geometrically irreducible of dimension D. By Lang-Weil, there exists a real

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[2]) In $\mathbf{G}_m/\mathbf{F}_p$, there are approximately p^i/i closed points of given norm p^i; the exact number always lies within $2p^{i/2}/i$ of p^i/i. So the number of closed points of norm at most p^n is at most $2\sum_{1\leq i\leq n} p^i/i$, while the number of closed points of norm p^{n+1} is at least $p^{n+1}/2(n+1)$. What is true (a calculus exercise) is that for any real $t > 8$, we have, for all $n \geq 1$, the inequality $t^{n+1}/(n+1) > 4\sum_{1\leq i\leq n} t^i/i$.

constant $C > 0$ such that for all integers $n \geq 0$, $\#X(\mathbf{F}_{q^n}) \geq q^{nD}(1 - C/q^{n/2})$. So for large n (e.g., large enough that $q^{n/2} \geq 2C$), we have $\#X(\mathbf{F}_{q^n}) \geq q^{nD}/2$. Similarly, for n large, the number of closed points of degree q^n is at least $q^{nD}/2n$. So we trivially have the estimate $\pi_X(t) \geq t^D/2\log_q(t)$ whenever t is a large power of q. So for all large t we have

$$\pi_X(t) \geq (t/q)^D/2\log_q(t/q) = q^{-D}t^D/(2(\log_q(t) - 1)) \geq q^{-D}t^D/3\log_q(t). \qquad \square$$

COROLLARY 4.2. *Let $V \subset X$ be a dense open set. Then Theorem* 3.4 *holds on X if and only if it holds on V.*

Proof. By Weyl's criterion, the packetwise equidistribution theorem is equivalent to the assertion that for every irreducible nontrivial representation Λ of K, we have the estimate

$$\sum_{\mathfrak{P} \in |X|,\ \mathrm{N}\mathfrak{P} \leq t} \mathrm{Trace}(\Lambda(\theta_{\mathfrak{P}})) = o(\pi_X(t)).$$

The assertion that it holds on V is that for each of these same Λ, we have the estimate

$$\sum_{\mathfrak{P} \in |V|,\ \mathrm{N}\mathfrak{P} \leq t} \mathrm{Trace}(\Lambda(\theta_{\mathfrak{P}})) = o(\pi_V(t)).$$

Each summand $\mathrm{Trace}(\Lambda(\theta_{\mathfrak{P}}))$ has absolute value at most $\dim(\Lambda)$. So the equivalence is immediate from the previous lemma, applied to the proper closed subscheme $Z := X \setminus V$ of X. $\square$

We will use this corollary as follows. Recall that by Pink's theorem, there is an open dense set $U \subset S$ such that for each closed point $\mathfrak{p}$ of U, we have

$$G_{geom,X_{\mathfrak{p}}} = G_{geom,X/S}.$$

By the corollary, it suffices to prove packetwise equidistribution on $V := f^{-1}(U)$. Thus we reduce to proving universally the packetwise equidistribution theorem for the situation $(X/S, \mathcal{F}, \iota)$ under hypothesis (H) and the additional hypothesis (AFG) ("all fibres good"):

Hypothesis (AFG) : for every closed point $\mathfrak{p} \in S$, $G_{geom,X_{\mathfrak{p}}} = G_{geom,X/S}$.

5. A STRONGER EQUIDISTRIBUTION THEOREM, WHEN BOTH HYPOTHESES (H) AND (AFG) HOLD

In this section, we suppose both hypotheses (H) and (AFG) hold. We arrange the closed points $\mathfrak{P}$ into packets $P(\mathfrak{p}, n)$ labeled by the underlying closed point $\mathfrak{p}$ of S and by the integer $n \geq 1$ which is the degree of $\mathbf{F}_{\mathfrak{P}}/\mathbf{F}_{\mathfrak{p}}$. In other words, $P(\mathfrak{p}, n)$ consists of the closed points of $X_{\mathfrak{p}}$ whose degree over $\mathbf{F}_{\mathfrak{p}}$ is n.

For each nonempty packet, denote by $\mu(P(\mathfrak{p}, n))$ the measure on $K^{\#}$ defined by

$$\mu(P(\mathfrak{p}, n)) := (1/\#P(\mathfrak{p}, n)) \sum_{\mathfrak{P} \in P(\mathfrak{p}, n)} \delta_{\theta_{\mathfrak{P}}} .$$

THEOREM 5.1 (Packet by Packet Equidistribution). *Suppose that both hypotheses (H) and (AFG) hold. Let $(\mathfrak{p}_i, n_i)$ be a sequence of pairs consisting of a closed point of S and a strictly positive integer. Suppose that $\mathbf{N}\mathfrak{p}_i^{n_i}$ tends archimedeanly to ∞. Then the sequence of measures $\mu(P(\mathfrak{p}_i, n_i))$ tends weakly to the induced "Haar measure" of total mass one on $K^{\#}$.*

PROPOSITION 5.2. *When hypotheses (H) and (AFG) hold, Theorem 5.1 implies Theorem 3.4.*

Proof. For each nonempty packet, and each irreducible nontrivial Λ, we consider the fraction

$$N(\mathfrak{p}, n, \Lambda)/\#P(\mathfrak{p}, n)$$

with

$$N(\mathfrak{p}, n, \Lambda) := \sum_{\mathfrak{P} \in P(\mathfrak{p}, n)} \mathrm{Trace}(\Lambda(\theta_{\mathfrak{P}})) .$$

Theorem 5.1 is, by the Weyl criterion, the assertion that for each irreducible nontrivial representation Λ of K, the fractions $N(\mathfrak{p}, n, \Lambda)/\#P(\mathfrak{p}, n)$ tend to 0 as $\mathbf{N}\mathfrak{p}^n$ tends to ∞. What we must prove is that for each irreducible nontrivial Λ, the fractions

$$\frac{\sum_{(\mathfrak{p}, n) \text{ with } \mathbf{N}\mathfrak{p}^n \leq t} N(\mathfrak{p}, n, \Lambda)}{\sum_{(\mathfrak{p}, n) \text{ with } \mathbf{N}\mathfrak{p}^n \leq t} \#P(\mathfrak{p}, n)}$$

tend to 0 as t tends archimedeanly to ∞.

It suffices to apply the following elementary lemma, whose proof is left to the reader.

LEMMA 5.3. *Let a_n/b_n be a sequence of fractions with $a_n \in \mathbf{C}$ and $b_n \in \mathbf{R}_{>0}$. Suppose that $|a_n/b_n| \to 0$ as n tends to ∞, and that $\sum_{i\leq n} b_i$ tends to ∞ as n tends to ∞. Then the fractions $\frac{\sum_{i\leq n} a_i}{\sum_{i\leq n} b_i}$ tend to 0 as n tends to ∞.* $\square$

We now turn to the proof of Theorem 5.1. In [Ka-Sar, 9.6.10], ratios similar to the ratios $N(\mathfrak{p}, n, \Lambda)/\#P(\mathfrak{p}, n)$ are considered. [The assumption "(9.3.5.1)" stated there is used only to insure that $G_{arith,X} \subset \mathbf{G}_m G_{geom,X/S}$.] That result is stated in terms of conjugacy classes $\theta_{k,s,\alpha_s,x}$ attached to finite-field valued points of X, say $x \in X(k)$, partitioned according to the underlying k-valued point s of S, with α_s there our $\alpha_\mathfrak{p}^{deg(\mathbf{F}_s/\mathbf{F}_\mathfrak{p})}$. It is proven there that there exist positive integer constants $A(X/S)$ and $C(X/S, \mathcal{F})$ such that for each nonempty packet $P(\mathfrak{p}, n)$ with $\mathbf{N}\mathfrak{p}^n \geq 4A(X/S)^2$, and $\mathbf{F}_{\mathfrak{p},n}/\mathbf{F}_\mathfrak{p}$ the extension of degree n, we have

$$\#X_\mathfrak{p}(\mathbf{F}_{\mathfrak{p},n}) \geq \mathbf{N}\mathfrak{p}^{nd}/2$$

and

$$\Big|\frac{1}{\#X_\mathfrak{p}(\mathbf{F}_{\mathfrak{p},n})} \sum_{x\in X_\mathfrak{p}(\mathbf{F}_{\mathfrak{p},n})} \mathrm{Trace}(\Lambda(\theta_{\mathbf{F}_{\mathfrak{p},n},\mathfrak{p},\alpha_\mathfrak{p}^n,x}))\Big|$$
$$\leq 2C(X/S, \mathcal{F}) \dim(\Lambda)/\mathbf{N}\mathfrak{p}^{n/2}.$$

We now have only to turn this into an estimate for our fractions $N(\mathfrak{p}, n, \Lambda)/\#P(\mathfrak{p}, n)$. To do this, we use the fact that closed points $\mathfrak{P}$ of $X_\mathfrak{p}$ of degree n over $\mathbf{F}_\mathfrak{p}$ are simply the orbits of $\mathrm{Gal}(\mathbf{F}_\mathfrak{p}^{sep}/\mathbf{F}_\mathfrak{p})$ of length n in $X_\mathfrak{p}(\mathbf{F}_{\mathfrak{p},n})$, and that each of the n points in such an orbit gives rise to the same conjugacy class $\theta_\mathfrak{P}$. The number of points of $X_\mathfrak{p}(\mathbf{F}_{\mathfrak{p},n})$ which lie in $X_\mathfrak{p}(\mathbf{F}_{\mathfrak{p},n/r})$ for some divisor $r \geq 2$ of n is at most $2(1 + A(X/S))\mathbf{N}\mathfrak{p}^{nd/2}$. So the sum

$$\sum_{x\in X_\mathfrak{p}(\mathbf{F}_{\mathfrak{p},n})} \mathrm{Trace}(\Lambda(\theta_{\mathbf{F}_{\mathfrak{p},n},\mathfrak{p},\alpha_\mathfrak{p}^n,x}))$$

differs from the sum

$$nN(\mathfrak{p}, n, \Lambda)$$

by an error bounded by $2(1 + A(X/S))\mathbf{N}\mathfrak{p}^{nd/2}\dim(\Lambda)$. And $\#X_\mathfrak{p}(\mathbf{F}_{\mathfrak{p},n})$ differs from $n\#P(\mathfrak{p}, n)$ by at most $2(1 + A(X/S))\mathbf{N}\mathfrak{p}^{nd/2}$. We conclude that for $\mathbf{N}\mathfrak{p}^n$ sufficiently large, we have the estimate

$$(*) \quad |N(\mathfrak{p}, n, \Lambda)/\#P(\mathfrak{p}, n)| \leq 4C(X/S, \mathcal{F}) \dim(\Lambda)/\mathbf{N}\mathfrak{p}^{n/2}.$$

This concludes the proof of Theorem 5.1. $\square$

Here is an application of Theorem 5.1. For each integer $N > 1$, denote by $P(N)$ the packet consisting of all the closed points of norm N. For each N such that $P(N)$ is nonempty, denote by $\mu(P(N))$ the measure on $K^\#$ defined by

$$\mu(P(N)) := (1/\#P(N)) \sum_{\mathfrak{P} \in P(N)} \delta_{\theta_{\mathfrak{P}}}.$$

THEOREM 5.4 (Norm by Norm Equidistribution). *Suppose that both hypotheses (H) and (AFG) hold. Then as N tends to ∞ over norms of closed points of X, the measures $\mu(P(N))$ on $K^\#$ tend weakly to the induced "Haar measure" of total mass one on $K^\#$.*

Proof. Indeed, the packet $P(N)$ is the disjoint union of those packets $P(\mathfrak{p}, n)$ for which $\mathbf{N}\mathfrak{p}^n = N$. For N large and $P(N)$ nonempty, the estimate $(*)$ shows that for each nontrivial irreducible Λ we have the estimate

$$|(1/\#P(N)) \sum_{\mathbf{N}\mathfrak{P}=N} \mathrm{Trace}(\Lambda(\theta_{\mathfrak{P}}))| \le 4C(X/S, \mathcal{F}) \dim(\Lambda)/N^{1/2}. \qquad \square$$

As another application of Theorem 5.1 and Lemma 5.3, we have the following variant of the packetwise equidistribution theorem.

THEOREM 5.5 (Partial Packetwise Equidistribution). *Suppose that both hypotheses (H) and (AFG) hold. Let $(\mathfrak{p}_i, n_i)$ be a sequence of pairs consisting of a closed point of S and a strictly positive integer. Suppose that $\mathbf{N}\mathfrak{p}_i^{n_i}$ tends archimedeanly to ∞. Then as $d \to \infty$ the sequence of measures on $K^\#$*

$$\nu_d := \frac{1}{\sum_{1 \le i \le d} \#P(\mathfrak{p}_i, n_i)} \sum_{1 \le i \le d} \sum_{\mathfrak{P} \in P(\mathfrak{p}_i, n_i)} \delta_{\theta_{\mathfrak{P}}}$$

tends weakly to the induced "Haar measure" of total mass one on $K^\#$.

For example, we could take any sequence $(\mathfrak{p}_i, 1)$ with closed points of S whose norms tend archimedeanly to ∞. If S has generic characteristic zero, we could further restrict to using only closed points with prime fields as residue fields (cf. [Se-$N_X(p)$, 9.1.4]), and we could choose any infinite subset of these if we wish.

REMARK 5.6. Even when X has generic characteristic zero, it will not be the case in general that the collection of closed points in $\cup_i P(\mathfrak{p}_i, n_i)$ satisfies classical equidistribution. For example, we might choose the sequence $\mathbf{N}\mathfrak{p}_i^{n_i}$

to be so lacunary that at each approximating step we add on more points than we had before (i.e. $\#P(\mathfrak{p}_{d+1}, n_{d+1}) \geq \sum_{1 \leq i \leq d} \#P(\mathfrak{p}_i, n_i)$).

6. A VARIANT: "ELIMINATING" S

In this section, we suppose hypothesis (H) holds, and we make an additional hypothesis (Sm) on the base S.

(1) If S has generic characteristic zero, we suppose given a number field L and an integer N such that S is a smooth $\mathcal{O}_L[1/N\ell]$-scheme with geometrically connected fibres of some common dimension $e \geq 0$.

(2) If S has characteristic $p > 0$, we suppose given a finite field $\mathbf{F}_q$ such that S is a smooth, geometrically connected $\mathbf{F}_q$-scheme of dimension $e \geq 0$.

For ease of reference, we denote by S_0 the scheme $\mathrm{Spec}(\mathcal{O}_L[1/N\ell])$, respectively $\mathrm{Spec}(\mathbf{F}_q)$, in the two cases.

LEMMA 6.1. *Under hypotheses (H) and (Sm) on $(X/S, \mathcal{F}, \iota)$, the situation $(X/S_0, \mathcal{F}, \iota)$ satisfies hypothesis (H).*

Proof. Denote by $\overline{\eta_0}$ the geometric generic point of S_0. Then $\overline{\eta}$, the geometric generic point of S, lies over $\overline{\eta_0}$, so the homomorphism

$$\pi_1(X_{\overline{\eta}}, \overline{\xi}) \to \pi_1(X, \overline{\xi})$$

factors as

$$\pi_1(X_{\overline{\eta}}, \overline{\xi}) \to \pi_1(X_{\overline{\eta_0}}, \overline{\xi}) \to \pi_1(X, \overline{\xi}).$$

Thus the Zariski closure groups are related by

$$G_{geom,X/S} \subset G_{geom,X/S_0} \subset G_{arith,X} \subset \mathbf{G}_m G_{geom,X/S} \subset \mathbf{G}_m G_{geom,X/S_0},$$

the penultimate inclusion by hypothesis (H). Thus

$$G_{arith,X} \subset \mathbf{G}_m G_{geom,X/S_0},$$

as required. □

The group $G_{geom,X/S_0}$ is semisimple, and being caught between $G_{geom,X/S}$ and $\mathbf{G}_m G_{geom,X/S}$ must be $\mu_n G_{geom,X/S}$ for some integer $n \geq 1$.

When we apply Theorem 3.4 to this $(X/S_0, \mathcal{F}, \iota)$ situation, we are getting packetwise equidistribution of conjugacy classes, call them $\theta^0_{\mathfrak{P}}$, in the space $(\mu_n K)^\#$ of conjugacy classes of the group $\mu_n K$, instead of packetwise equidistribution of conjugacy classes $\theta_{\mathfrak{P}}$ in the space $K^{\#}$ of conjugacy classes of the group K.

To see how the two results differ, start with $(X/S, \mathcal{F}, \iota)$, and a lisse rank one $\overline{\mathbf{Q}_\ell}$-sheaf $\mathcal{L}$ on S. Denote by $\mathcal{L}_1$ the pullback of $\mathcal{L}$ to X. Now consider the situation $(X/S, \mathcal{F} \otimes \mathcal{L}_1, \iota)$. From the point of view of the packetwise equidistribution theorem for X/S, nothing has changed, since $\mathcal{L}_1$ is trivial on $X_{\overline{\eta}}$. We simply do not see $\mathcal{L}_1$. However, from the point of view of X/S_0, we might very well see it.

Here is a concrete example. Choose an integer $n \geq 1$, and denote by R the cyclotomic ring $R := \mathbf{Z}[\zeta_n, 1/6n]$. Take $\ell = 2$,

$$S_0 = \mathrm{Spec}(R),$$

$$S = \mathrm{Spec}(R[g_3, 1/g_3]) = \mathbf{G}_m/R,$$

$$X = \mathrm{Spec}(R[g_2, g_3, 1/g_3, 1/(g_2^3 - 27g_3^2)]).$$

Over X we have the Weierstrass family $\mathcal{W}/X$, affine equation $y^2 = 4x^3 - g_2x - g_3$. For $\mathcal{F}$ on X its H^1 along the fibres, $G_{geom,X/S}$ is $SL(2)$ (indeed, on each geometric fibre of X/S, the j-invariant is nonconstant). Since $\mathcal{F}$ is of rank 2, we certainly have $G_{arith,X} \subset \mathbf{G}_m G_{geom,X/S} = GL(2)$. Now take for $\mathcal{L}$ on $S = \mathbf{G}_m/R$ a Kummer sheaf $\mathcal{L}_{\chi(g_3)}$ with χ a character of order n. Then $\mathcal{F} \otimes \mathcal{L}_1$ on X has $G_{geom,X/S_0} = \mu_n SL(2)$.

7. Another variant

In the situation of the previous section, $(X/S/S_0, \mathcal{F}, \iota)$ with $(X/S, \mathcal{F}, \iota)$ satisfying hypotheses (H) and (Sm), there is a stronger hypothesis we could impose, hypothesis (H_1):

$$\text{Hypothesis } (H_1): G_{geom,X/S} = G_{arith,X},$$

in other words $\rho(\pi_1(X)) \subset G_{geom,X/S}$. In this case, the inclusions $G_{geom,X/S} \subset G_{geom,X/S_0} \subset G_{arith,X}$ show that

$$G_{geom,X/S} = G_{geom,X/S_0}.$$

So for packetwise equidistribution on X of all the $\theta_{\mathfrak{P}}$'s there is no difference between the result for $(X/S, \mathcal{F}, \iota)$ and the result for $(X/S_0, \mathcal{F}, \iota)$. There is however the difference that if hypothesis (AFG) holds for both $(X/S, \mathcal{F}, \iota)$ **and** for $(X/S_0, \mathcal{F}, \iota)$, then in Theorems 5.1 and 5.5, we get to select finer packets in the X/S context than in the X/S_0 context.

8. Some examples

In this section, we illustrate the general theory with a few concrete examples. We begin with curves. Fix a genus $g \geq 1$, and a monic polynomial $f_{2g}(X) \in \mathbf{Z}[X]$ of degree $2g$ whose discriminant Δ is nonzero. We consider the one-parameter family of genus g curves, parameter λ, given in affine form as

$$Y^2 = f_{2g}(X)(X - \lambda)$$

over the parameter space $X := \mathrm{Spec}(\mathbf{Z}[\lambda, 1/(2f_{2g}(\lambda)\Delta)])$, say $\pi\colon \mathcal{C} \to X$. On X we have the lisse sheaf $\mathcal{F} := R^1\pi_!\overline{\mathbf{Q}_2}$, which is pure of weight one. We take $S := \mathrm{Spec}(\mathbf{Z}[1/(2\Delta)])$. Here one has [Ka-Sar, 10.1.16] $G_{geom,X/S} = Sp(2g)$ and $G_{arith,X} = GSp(2g)$, so hypothesis (H) holds. In this example, (AFG) also holds.

Here is another curve example. Begin with a monic polynomial $f_{2g+1}(X) \in \mathbf{Z}[X]$ of degree $2g+1$. Denote by δ the gcd of the coefficients of the second derivative f''_{2g+1} of f_{2g+1}. Consider the two-parameter family of genus g curves, parameters A, B, given in affine form as

$$Y^2 = f_{2g+1}(X) + AX + B$$

over the parameter space $X := \mathrm{Spec}(\mathbf{Z}[A, B, 1/(2\delta\Delta)])$, for $\Delta \in \mathbf{Z}[A, B]$ the discriminant of $f_{2g+1}(X)+AX+B$ (which is nonzero, cf. [Ka-ACT, 3.5]). Here we may take $S := \mathrm{Spec}(\mathbf{Z}[1/(2\delta)])$, or we may take $S := \mathrm{Spec}(\mathbf{Z}[A, 1/(2\delta)])$. For either choice of S, one has [Ka-Sar, 10.3.1, 10.3.2] $G_{geom,X/S} = Sp(2g)$ and $G_{arith,X} = GSp(2g)$, so hypothesis (H) holds.

Here are some hypersurface examples, cf. [Ka-Sar, 10.4.9] and [De-Weil II, 4.4.1]. Take integers $d \geq 3$ and $n \geq 1$, with $(n, d) \neq (2, 3)$, and consider the universal family of smooth projective hypersurfaces of dimension n and degree d, say $\pi\colon \mathcal{X}_{n,d} \to \mathcal{H}_{n,d}$. Fix a prime ℓ, take $X := \mathcal{H}_{n,d}[1/\ell]$ and $S := \mathrm{Spec}(\mathbf{Z}[1/\ell])$. When n is odd, we take $\mathcal{F}$ on $X[1/\ell]$ to be $R^n\pi_\star\overline{\mathbf{Q}_\ell}$, which is lisse of rank

$$\mathrm{prim}(n, d) := ((d-1)/d)((d-1)^{n+1} - (-1)^{n+1})$$

and pure of weight n. Here $G_{geom,X/S} = Sp(\mathrm{prim}(n, d))$, $G_{arith,X} = GSp(\mathrm{prim}(n, d))$, and hypotheses (H) and (AFG) both hold. When n is even, we take $\mathcal{F}$ on $X[1/\ell]$ to be $Prim^n\pi_\star\overline{\mathbf{Q}_\ell}(n/2)$ (cf. [Ka-Sar, 11.4.8] for the definition of $Prim^n$), which is lisse of rank $\mathrm{prim}(n, d)$ and pure of weight zero. We have $G_{geom,X/S} = G_{arith,X} = O(\mathrm{prim}(n, d))$. Here hypotheses (H_1) and (AFG) both hold.

9. Examples where Hypothesis (*H*) fails

Let us begin with S a noetherian connected scheme of finite type over $\mathbf{Z}[1/\ell]$, and $\mathcal{F}_0$ a lisse $\overline{\mathbf{Q}_\ell}$ sheaf on S of rank $n \geq 2$ which is ι-pure of weight zero, such that in the representation ρ_0 of $\pi_1(S)$ which $\mathcal{F}_0$ "is", the image of $\pi_1(S)$ is not abelian[3]). Now take **any** X/S which is smooth, with geometrically connected fibres of some dimension $d \geq 1$, and take $\mathcal{F}$ on X to be the pullback of $\mathcal{F}_0$. Then on each geometric fibre of X/S, $\mathcal{F}$ is constant, hence $G_{geom,X/S}$ is the trivial subgroup of $GL(n)$. But the group $G_{arith,X}$ is the group $G_{arith,S} \subset GL(n)$ (because $\mathcal{F}$ was the pullback from S of $\mathcal{F}_0$), and this group is not abelian. So Hypothesis (*H*) does not hold. While Jordan-semisimplified Frobenius classes from $\pi_1(S)$ may or may not be equidistributed (in some or all of the various senses of equidistribution discussed above) in a compact form[4]) of $G_{arith,S}$, the moral of this example is that invoking an X/S and pulling back to X will never help us.

Here is another example where Hypothesis (*H*) does not hold. Take for S_0 the Spec of $\mathbf{Z}[1/2\ell]$, and for X/S_0 the product $\mathbf{G}_m \times \mathbf{G}_m$, with coordinates x, y. Take χ to be the quadratic character, and take $\mathcal{F}$ on X to be the direct sum of the two Kummer sheaves

$$\mathcal{F} := \mathcal{L}_{\chi(x)} \oplus \mathcal{L}_{\chi(xy)}\,.$$

Then

$$G_{geom,X/S_0} = G_{arith,X} = \mu_2 \times \mu_2\,,$$

and Hypothesis (*H*) holds for X/S_0. However, if we take S to be $\mathbf{G}_m/S_0$ and view X as lying over S by the second projection, then $G_{geom,X/S}$ is the group μ_2, embedded diagonally in $\mu_2 \times \mu_2$. So for X/S, Hypothesis (*H*) does not hold.

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[3]) For example, take $\ell = 776887$, take S to be the Spec of $\mathbf{Z}[1/776887]$, and take $\mathcal{F}_0$ as follows. The polynomial $x^7 - x - 1$ over $\mathbf{Q}$ has Galois group the full symmetric group S_7, and the discriminant of this polynomial is -776887. We take for $\mathcal{F}_0$ the lisse $\overline{\mathbf{Q}_\ell}$ sheaf on S of rank 7 incarnating this representation $\rho_0 \colon \pi_1(S) \twoheadrightarrow S_7 \subset GL(7)$ (here the inclusion $S_7 \subset GL(7)$ is by the usual permutation action on the coordinates).

[4]) To be able to speak of a compact form of $G_{arith,S}$, we must also assume that $\mathcal{F}_0$ is a completely reducible representation of $\pi_1(S)$; then the group $G_{arith,S}$ will be reductive, and hence will have a compact form.

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Nicholas M. Katz

Princeton University
Mathematics
Fine Hall
NJ 08544-1000
U.S.A.
e-mail: nmk@math.princeton.edu