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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **56 (2010)**

PDF erstellt am: **28.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-283521

L'Enseignement Mathématique (2) 56 (2010), 287–313

DEFORMATIONS ALONG SUBSHEAVES

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ABSTRACT. Let $f: Y \to X$ be a morphism of complex manifolds, and assume that Y is compact. Let $\mathcal{F} \subseteq T_X$ be a subsheaf which is closed under the Lie bracket. The present paper contains an elementary and very geometric argument to show that all obstructions to deforming f along the sheaf $\mathcal{F}$ lie in $H^1(Y, \mathcal{F}_Y)$, where $\mathcal{F}_Y \subseteq f^*(T_X)$ is the image of $f^*(\mathcal{F})$ under the pull-back of the inclusion map. Special cases of this result include Miyaoka's theory of deformation along a foliation, Keel-McKernan's logarithmic deformation theory and deformations with fixed points.

CONTENTS

*) Stefan Kebekus was supported in part by the DFG-Forschergruppe "Classification of Algebraic Surfaces and Compact Complex Manifolds". Some parts of this paper were worked out during a visit of Kebekus to the University of Michigan at Ann Arbor. He would like to thank Mircea Mustaţă for the invitation and for a number of discussions. The work on this paper was finished while Kebekus visited the 2009 Special Year in Algebraic Geometry at the Mathematical Sciences Research Institute, Berkeley. He would like to thank the MSRI for support. Daniel Lohmann was supported in part by the DFG-Graduiertenkolleg "Global Structures in Geometry and Analysis".

1. Introduction and main results

1.A Introduction

Let $f \colon Y \to X$ be a morphism of complex manifolds and assume that Y is compact. We aim to deform f, keeping X and Y fixed. More precisely, given an *infinitesimal deformation* of f, say $\sigma \in H^0\big(Y, f^*(T_X)\big)$, we ask if σ is *effective*, i.e., if σ comes from a deformation of f.

It is a classical result that any infinitesimal deformation is effective if the associated obstruction space vanishes. We refer to [Hor73], or to [Kol96, Chap. 1] for a thorough discussion of the algebraic case.

THEOREM 1.1. *If $H^1\big(Y, f^*(T_X)\big) = \{0\}$, then any infinitesimal deformation of f is effective.*

Theorem 1.1 is not sharp however. There are many examples of infinitesimal deformations that are effective even though $h^1\big(Y, f^*(T_X)\big)$ is large. In these cases, it is often possible to find a geometric reason that explains the behavior. Here, we consider the geometric context where there is a subsheaf $\mathcal{F} \subseteq T_X$, and where $\sigma \in H^0\big(Y, f^*(T_X)\big)$ is an infinitesimal deformation along $\mathcal{F}$, i.e., where σ is in the image of the natural map

$$H^0\big(Y, f^*(\mathcal{F})\big) \to H^0\big(Y, f^*(T_X)\big).$$

If $\mathcal{F}$ is closed under the Lie bracket, we show that an analogue of Theorem 1.1 holds for deformations along $\mathcal{F}$.

The proof of our main result, Theorem 1.5, is completely elementary and does not use any of the sophisticated methods of deformation theory. The methods also illustrate the proof of Theorem 1.1.

1.B Main result

In order to formulate the main results precisely in Theorem 1.5 below, recall a few standard definitions and notation used in the discussion of deformations.

DEFINITION 1.2. A *deformation of f* is a holomorphic mapping $F \colon \Delta \times Y \to X$ whose restriction to $\{0\} \times Y \cong Y$ equals f. Here $\Delta \subset \mathbf{C}$ is a disk centered about 0.

NOTATION 1.3. If F is a deformation and $t \in \Delta$ any number, we often write $F_t \colon Y \to X$ for the obvious restriction of F to $\{t\} \times Y \cong Y$. Given a point $y \in Y$, we can consider the curve

$$F_y \colon \Delta \to X, \quad t \mapsto F(t, y).$$

Given $t \in \Delta$ and taking derivatives in t for all y, this gives a section

$$\sigma_{F,t} \in H^0\big(Y, (F_t)^*(T_X)\big),$$

called *velocity vector field at time* t. For $t = 0$, we obtain a section $\sigma_{F,0} \in H^0\big(Y, f^*(T_X)\big)$. Elements of $H^0\big(Y, f^*(T_X)\big)$ are thus called *initial velocity vector fields* or *first order infinitesimal deformations of* f.

DEFINITION 1.4. A first order infinitesimal deformation $\sigma \in H^0\big(Y, f^*(T_X)\big)$ is *effective* if there exists a deformation F with $\sigma = \sigma_{F,0}$.

With this notation, the main result of the present paper is formulated as follows.

THEOREM 1.5 (Deformation along an involutive subsheaf). *Let $f \colon Y \to X$ be a morphism of complex manifolds and assume that Y is compact. Let $\mathcal{F} \subseteq T_X$ be a subsheaf of $\mathcal{O}_X$-modules which is closed under the Lie bracket, let $\mathcal{F}_Y \subseteq f^*(T_X)$ be the image of $f^*(\mathcal{F})$ under the pull-back of the inclusion map, and let*

$$\sigma \in H^0\big(Y, \mathcal{F}_Y\big) \subseteq H^0\big(Y, f^*(T_X)\big)$$

be a first order infinitesimal deformation of the morphism f that comes from $\mathcal{F}$.

If $H^1\big(Y, \mathcal{F}_Y\big) = \{0\}$, then there exists a deformation F of f such that $\sigma = \sigma_{F,0}$, and such that for all times $t \in \Delta$ the section $\sigma_{F,t}$ is in the image of

$$H^0\big(Y, (F_t)^*(\mathcal{F})\big) \to H^0\big(Y, (F_t)^*(T_X)\big). \tag{1.5.1}$$

NOTATION 1.6. If F is any deformation of f such that (1.5.1) holds for all t, we say that F is a deformation *along the sheaf* $\mathcal{T}$.

REMARK 1.7. The subsheaf $\mathcal{F} \subseteq T_X$ need not be a foliation because $\mathcal{F}$ need not be saturated in T_X. We recall a few special cases of Theorem 1.5 that we have found in the literature:

(1.7.1) *Foliations*. The case where $\mathcal{F}$ is an algebraic foliation is studied in Miyaoka's theory of deformation along an algebraically defined foliation, [Miy87, MP97].

(1.7.2) *Logarithmic tangent sheaves*. The case where X contains a reduced divisor D and $\mathcal{F} = T_X(-\log D)$ appears in Keel and McKernan's work on the Miyanishi conjecture, [KMc99, Sect. 5].

(1.7.3) *Deformation with fixed points*. A variant of the case where $\mathcal{F} = T_X \otimes \mathcal{J}_p$ is the tangent bundle twisted with the ideal sheaf of a point p is used in Mori's Bend-and-Break technique.

1.C Outline of the paper

In Section 2, we recall the definition of jet bundles on a complex manifold X and recall their main properties. The language of jets makes it easy to discuss n-th order deformations of a given morphism, and gives an elementary way to construct classes in $H^1\big(Y, f^*(T_X)\big)$ that are obstructions to extending n-th order deformations to $(n+1)$-th order. We illustrate these concepts by reproducing Horikawa's proof of Theorem 1.1 in the language of jets, referring to Artin's paper [Art68] for the necessary convergence results.

In Section 3, we outline the proof of Theorem 1.5, explain the main strategy and motivate two sets of problems which are discussed in Sections 4 and 5 before completing the proof of Theorem 1.5 in Section 6.

Section 4 concerns the relation between vector fields and higher order jets of the integral curves they define. Given two vector fields D_1 and D_2 on X with integral curves γ_1 and γ_2, we are interested in expressing the difference of higher order terms in the power series expansions of the γ_i in terms of iterated Lie brackets involving D_1 and D_2.

In Section 5, we discuss an elementary generalization of the classical Frobenius Theorem of Differential Geometry, where the Lie-closed subsheaf $\mathcal{F} \subseteq T_X$ is not necessarily a foliation. This will allow us to construct local analytic subspaces of the Douady space $\mathrm{Hom}(Y, X)$ which locally parametrize deformations along the sheaf $\mathcal{F}$.

1.D Acknowledgments

A first version of Theorem 1.5 and the elementary proof of Theorem 1.1 are contained in the diploma thesis of Daniel Lohmann and Stavros Kousidis, respectively. Both thesis projects were supervised by Stefan Kebekus.

The authors would like to thank Guido Kings for discussions, and the anonymous referee for a careful review, for suggestions that helped to remove a projectivity assumption, and for bringing Horikawa's paper to our attention.

2. Jet bundles and deformations of morphisms

In Sections 2.A–2.C we recall the definition and briefly discuss the main properties of jet bundles of a complex manifold, which are higher order generalizations of the tangent bundle. Jet bundles are then used in Section 2.D to describe higher-orderinfinitesimal deformations of morphisms. To illustrate the use of jets in deformation theory, we end this chapter with a short and very transparent proof of the classical Theorem 1.1.

REMARK 2.1. There are two notions of "jet bundle" found in the literature. In this paper, an "n-jet" is an n-th order curve germ. This notion was, originally introduced in slightly higher generality in real geometry by Ehresmann, cf. [Arn88, Chapt. 6.29C].

Other authors use the word "n-jet" to denote an n-th order germ of a section in a given vector bundle. This notion is found, e.g., in the work of Kumpera-Spencer on Lie equations, [KS72, Chap. 1].

NOTATION 2.2. If X and Y are any two complex spaces where Y is compact, we denote the Douady space of morphisms from Y to X by $\mathrm{Hom}(Y, X)$. Like the Hom-scheme of algebraic geometry, the Douady space of morphisms represents a functor and is therefore uniquely determined by its universal properties. We refer to [CP94, Sect. 2] for a brief overview and for further references.

The reader who is content with algebraic morphisms of projective varieties is free to use the Hom-scheme instead of the Douady space throughout this paper.

2.A Tangent bundles

Let X be a complex manifold. Before discussing jet bundles of arbitrary order, we recall two equivalent standard constructions of the tangent bundles for the reader's convenience.

CONSTRUCTION 2.3. As a manifold, the tangent bundle T_X is the set of equivalence classes of germs of arcs $\Delta \to X$, under the equivalence relation that $\tau \sim \sigma$ if they agree to first order. Coordinate charts on X induce coordinate charts on T_X in the obvious canonical manner, and the map $\tau \mapsto \tau(0)$ induces a canonical morphism $\pi\colon T_X \to X$.

CONSTRUCTION 2.4. As a complex space or scheme, the tangent bundle is defined as $T_X := \mathrm{Hom}\big(\mathrm{Spec}\,\mathbf{C}[\varepsilon]/(\varepsilon^2), X\big)$, where $\mathrm{Spec}\,\mathbf{C}[\varepsilon]/(\varepsilon^2)$ denotes the double point on the affine line. The obvious map $\mathbf{C}[\varepsilon]/(\varepsilon^2) \to \mathbf{C}$ induces a canonical morphism $\pi\colon T_X \to X = \mathrm{Hom}(\mathrm{Spec}\,\mathbf{C}, X)$.

Using either construction, an elementary computation immediately gives the following

FACT 2.5. *The tangent bundle T_X of a complex manifold X has the structure of a vector bundle over X.*

Local coordinates on $U \subseteq X$ induce vector bundle coordinates on $\pi^{-1}(U) \subseteq T_X$. More precisely, if $U \subseteq X$ is a coordinate neighborhood, and γ is a germ of an arc $\gamma\colon \Delta \to U$, described in U-coordinates as

$$\gamma(t) = \vec{x}_0 + \vec{x}_1 \cdot t + (\textit{higher-order terms}),$$

then the associated point of T_X has $\pi^{-1}(U)$-coordinates $(\vec{x}_0, \vec{x}_1) \in U \times \mathbf{C}^{\dim X}$.

2.B JET BUNDLES

In complete analogy with Constructions 2.3–2.4, the jet bundle of a complex manifold X can be defined in one of the following equivalent ways.

CONSTRUCTION 2.6. As a manifold, the n-th jet bundle $\mathrm{Jet}^n(X)$ is the set of equivalence classes of germs of arcs $\Delta \to X$, under the equivalence relation that $\tau \sim \sigma$ if they agree to n-th order. Coordinate charts on X induce coordinate charts on $\mathrm{Jet}^n(X)$ in the obvious canonical manner, and for any $m \leq n$ the restriction of arcs to m-th order induces a canonical morphism $\pi_{n,m}\colon \mathrm{Jet}^n(X) \to \mathrm{Jet}^m(X)$.

CONSTRUCTION 2.7. As a complex space or scheme the n-th jet bundle is defined as $\mathrm{Jet}^n(X) := \mathrm{Hom}\big(\mathrm{Spec}\,\mathbf{C}[\varepsilon]/(\varepsilon^{n+1}), X\big)$. For $m \leq n$, the truncation map $\mathbf{C}[\varepsilon]/(\varepsilon^{n+1}) \to \mathbf{C}[\varepsilon]/(\varepsilon^{m+1})$ induces a canonical morphism $\pi_{n,m}\colon \mathrm{Jet}^n(X) \to \mathrm{Jet}^m(X)$.

It is clear from the construction that $\mathrm{Jet}^0(X) \cong X$ and $\mathrm{Jet}^1(X) \cong T_X$. In complete analogy with Fact 2.5, an elementary computation in local coordinates shows the following

FACT 2.8. *Let X be a complex manifold and let $m \le n$ be any two integers. Then the following hold:*

(2.8.1) *The morphisms $\pi_{n,m}\colon \mathrm{Jet}^n(X) \to \mathrm{Jet}^m(X)$ are fiber bundles, locally trivial in Zariski topology with fibers isomorphic to $\mathbf{A}^{(n-m)\cdot \dim X}$. In general, the transition maps are neither linear nor affine, and $\pi_{n,m}$ is generally neither a vector bundle nor an affine bundle.*

(2.8.2) *Local coordinates on $U \subseteq X$ induce vector bundle coordinates on $\pi_{n,0}^{-1}(U) \subseteq \mathrm{Jet}^n(X)$, for all n. More precisely, if $U \subseteq X$ is a coordinate neighborhood, and γ is a germ of an arc $\gamma\colon \Delta \to U$, described in U-coordinates as*

$$\gamma(t) = \vec{v}_0 + \vec{v}_1 \cdot t + \cdots + \vec{v}_n \cdot t^n + (\textit{higher-order terms}),$$

then the associated point of $\mathrm{Jet}^n(X)$ has $\pi_{n,0}^{-1}(U)$-coordinates

$$(\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) \in U \times \mathbf{C}^{n\cdot \dim X},$$

with $\vec{x}_i = i! \cdot \vec{v}_i$. In particular, the coordinate $\vec{x}_i$ is computed in local coordinates as the i-th derivative, $\vec{x}_i = \gamma^{(i)}(0)$.

(2.8.3) *If $m = n-1$, the fiber bundle $\pi_{m+1,m}\colon \mathrm{Jet}^{m+1}(X) \to \mathrm{Jet}^m(X)$ has affine transition maps and is therefore an affine bundle.*

2.C AFFINE BUNDLES ASSOCIATED WITH JETS

We need to discuss the affine bundle structure of $\mathrm{Jet}^n(X) \to \mathrm{Jet}^{n-1}(X)$ in more detail. For that, we briefly recall the relevant properties of affine spaces.

By definition, any *affine space* A comes with a canonical vector space V, the *space of translations*, whose additive group V acts on A. The action map, often called *translation map* is usually denoted as follows:

$$+\colon V \times A \to A, \qquad (\vec{v}, a) \mapsto \vec{v} + a.$$

Given any $a \in A$, the natural map $V \to A$, $\vec{v} \mapsto \vec{v} + a$ is an isomorphism of complex manifolds. Consequently, given any two elements $a, b \in A$, there is a uniquely defined *difference vector* $\vec{v} \in V$, often denoted as $\vec{v} = a - b$, such that $\vec{v} + b = a$.

In complete analogy, any affine bundle $A \to B$ naturally comes with a vector bundle $\pi\colon V \to B$, the "bundle of translations". The translation maps on fibers glue to give a translation map

$$+\colon V \times_B A \to A.$$

Given any section $\sigma\colon B \to A$, the natural map $V \to A$, $\vec{v} \mapsto \vec{v} + \sigma(\pi(\vec{v}))$ is a fiber bundle isomorphism. Consequently, given any two sections $\sigma_1, \sigma_2\colon B \to A$, there is a uniquely defined *difference section,* $\tau\colon B \to V$, often denoted as $\tau = \sigma_1 - \sigma_2$, such that $\tau(b) + \sigma_2(b) = \sigma_1(b)$ for all $b \in B$.

For the affine bundle $\mathrm{Jet}^{n+1}(X) \to \mathrm{Jet}^n(X)$, the elementary computation used to prove Fact 2.9 immediately identifies the translation bundle.

FACT 2.9. *Let X be a complex manifold and let $n \geq 0$ be any number. Then the vector bundle V_n of translations associated with the affine bundle $\mathrm{Jet}^{n+1}(X) \to \mathrm{Jet}^n(X)$ is precisely the pull-back of the vector bundle T_X to $\mathrm{Jet}^n(X)$. In other words, $V_n = \pi_{n,0}^*(T_X)$.*

In the setup of Fact 2.9, if $\sigma_1, \sigma_2\colon X \to \mathrm{Jet}^{n+1}(X)$ are two sections that agree to n-th order, $\pi_{n+1,n} \circ \sigma_1 = \pi_{n+1,n} \circ \sigma_2$, then the difference is given by a section $\sigma_1 - \sigma_2 \in H^0\big(X, T_X\big)$. We will later need the following elementary generalization of this fact.

REMARK 2.10. If $f\colon Y \to X$ is a morphism of complex manifolds and if $\sigma_1, \sigma_2\colon Y \to f^* \mathrm{Jet}^{n+1}(X) = \mathrm{Jet}^{n+1}(X) \times_X Y$ are two sections in the pull-back bundles that agree to n-th order, $f^*(\pi_{n+1,n}) \circ \sigma_1 = f^*(\pi_{n+1,n}) \circ \sigma_2$, then the difference is given by a section $\sigma_1 - \sigma_2 \in H^0\big(Y, f^*(T_X)\big)$.

We end this section with a remark that shows how to compute the difference of jets in local coordinates. The (easy) proof is again left to the reader.

REMARK 2.11. If $U \subseteq X$ is a coordinate neighborhood, and if γ_1, $\gamma_2 \in \mathrm{Jet}^{n+1}(X)$ are two jets with $\pi_{n+1,n}(\gamma_1) = \pi_{n+1,n}(\gamma_2)$, represented in the induced coordinates on $\pi_{n+1,0}^{-1}(U) \subseteq \mathrm{Jet}^{n+1}(X)$ as

$$\gamma_1 = (\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_n, \vec{x}_{n+1,1}) \quad \text{and} \quad \gamma_2 = (\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_n, \vec{x}_{n+1,2}),$$

then the difference $\gamma_1 - \gamma_2$ is given by the tangent vector written in the induced coordinates on T_X as $\gamma_1 - \gamma_2 = (\vec{x}_0, \vec{x}_{n+1,1} - \vec{x}_{n+1,2}) \in T_X|_{\gamma_1(0)}$. If the base point $\gamma_1(0)$ is clear, we will often write

$$\gamma_1 - \gamma_2 = \vec{x}_{n+1,1} - \vec{x}_{n+1,2} \in T_X|_{\gamma_1(0)}.$$

2.D HIGHER-ORDER INFINITESIMAL DEFORMATIONS IN JET LANGUAGE

The following notion is the higher-order analogue of the infinitesimal deformation discussed in the introduction.

DEFINITION 2.12. Let $f\colon Y \to X$ be a morphism of complex manifolds. An *n-th order infinitesimal deformation of f* is a morphism

$$f_n\colon \operatorname{Spec} \mathbf{C}[\varepsilon]/(\varepsilon^{n+1}) \times Y \to X,$$

whose restriction to $Y \cong \operatorname{Spec} \mathbf{C} \times Y$ agrees with f.

It is clear from the universal property of the Douady space of morphisms that an n-th order infinitesimal deformation of f is the same as a morphism $\operatorname{Spec} \mathbf{C}[\varepsilon]/(\varepsilon^{n+1}) \to \operatorname{Hom}(Y,X)$ which maps the closed point to the point of $\operatorname{Hom}(Y,X)$ that represents f. For our purposes, however, the following description is more useful. It also shows that for $n=1$, Definition 2.12 and Notation 1.3 agree.

PROPOSITION 2.13. *To give an n-th order infinitesimal deformation of f, it is equivalent to give a section $Y \to f^* \operatorname{Jet}^n(X)$, where $f^* \operatorname{Jet}^n(X) := \operatorname{Jet}^n(X) \times_X Y$.*

Proof. It is clear from the universal property of $\operatorname{Hom}(Y,X)$ that to give an n-th order infinitesimal deformation of f, it is equivalent to give a morphism

$$\phi_n\colon Y \to \operatorname{Hom}\big(\operatorname{Spec} \mathbf{C}[\varepsilon]/(\varepsilon^{n+1}),\, X\big) = \operatorname{Jet}^n(X)$$

with $\pi_{n,0} \circ \phi_n = f$. By the universal property of the fiber product, this is the same as to give a section.

2.E APPLICATIONS TO DEFORMATIONS AND TO THEOREM 1.1

As an application of the methods and the language outlined in the previous sections, we reproduce in part Horikawa's proof of Theorem 1.1, referring to Artin's paper [Art68] for the necessary convergence results. More detailed computations are found in [Hor73].

The proof follows the common approach to first construct a *formal deformation* of f, which is then turned into a holomorphic solution. The existence of a formal solution is guaranteed by the following lemma which asserts that any n-th order infinitesimal deformation can be lifted to $(n+1)$-th order.

LEMMA 2.14. *In the setup of Theorem* 1.1, *let* $\sigma_n\colon Y \to f^*\,\mathrm{Jet}^n(X)$ *be any section. Then there exists a lifting to* $(n+1)$*-th order, i.e., a section* $\sigma_{n+1}\colon Y \to f^*\,\mathrm{Jet}^{n+1}(X)$ *making the following diagram commutative:*

(2.14.1)
$$\begin{tikzcd} & f^*\,\mathrm{Jet}^{n+1}(X) \arrow[d, "f^*(\pi_{n+1,n})"] \\ Y \arrow[ur, bend left, "\exists\,\sigma_{n+1}"] \arrow[r, "\sigma_n"] & f^*\,\mathrm{Jet}^n(X)\,. \end{tikzcd}$$

Proof. Since both $f^*\,\mathrm{Jet}^n(X)$ and $f^*\,\mathrm{Jet}^{n+1}(X)$ are locally trivial on Y, it is clear that liftings to $(n+1)$-th order always exist locally. More precisely, there exists a covering of Y with open sets $(U_\alpha)_{\alpha\in A}$ and there are sections $\sigma^\alpha_{n+1}\colon U_\alpha \to f^*\,\mathrm{Jet}^{n+1}(X)$ such that $f^*(\pi_{n+1,n})\circ\sigma^\alpha_{n+1} = \sigma_n|_{U_\alpha}$. We have seen in Remark 2.10 that for any $\alpha,\beta\in A$, the difference defines a section

$$\nu_{\alpha\beta} = \sigma^\alpha_{n+1}|_{U_\alpha\cap U_\beta} - \sigma^\beta_{n+1}|_{U_\alpha\cap U_\beta} \in H^0\big(U_\alpha\cap U_\beta, f^*(T_X)\big)\,.$$

The $\nu_{\alpha\beta}$ obviously satisfy the Čech cocycle condition and we obtain a cohomology class $(\nu_{\alpha\beta})\in H^1\big(Y, f^*(T_X)\big)$ which is zero by assumption.

Consequently, there are sections $\lambda_\alpha\in H^0\big(U_\alpha, T_X\big)$ with $\lambda_\alpha-\lambda_\beta=\nu_{\alpha\beta}$. If we set

$$\sigma'^\alpha_{n+1} := (-\lambda_\alpha) + \sigma^\alpha_{n+1}\,,$$

then σ'^α_{n+1} and σ'^β_{n+1} agree on $U_\alpha\cap U_\beta$ for any $\alpha,\beta\in A$ and therefore define a global section $\sigma_{n+1}\colon Y\to f^*\,\mathrm{Jet}^{n+1}(X)$ that lifts σ_n.

Proof of Theorem 1.1. Let $\sigma\in H^0\big(Y, f^*(T_X)\big)$ be any first order infinitesimal deformation. Choose a neighborhood U of the point $[f]\in\mathrm{Hom}\big(Y,X\big)$, and view U as a subset of $\mathbf{A}^n$, given by equations $U=\{f_1=\cdots=f_m=0\}$. With this notation, our aim is to find a holomorphic map $\widetilde{\sigma}\colon\Delta\to\mathbf{A}^n$ which agrees with σ to first order and satisfies $f_i\circ\widetilde{\sigma}=0$ for all i. By Michael Artin's result on solutions of analytic equations, [Art68, Thm 1.2], a holomorphic solution will exist if there is a formal solution to the problem.

Using Lemma 2.14 inductively, we can find a sequence $\sigma=\sigma_1,\sigma_2,\dots$ of liftings to arbitrary order, with $\pi_{n+1,n}\circ\sigma_{n+1}=\sigma_n$. If we view the σ_n as morphisms

$$\sigma_n\colon\ \mathrm{Spec}\,\mathbf{C}[\varepsilon]/(\varepsilon^{n+1})\to\mathrm{Hom}\big(Y,X\big)\,,$$

this defines a formal map

$$\widehat{\sigma}\colon\ \mathrm{Spec}\,\mathbf{C}[[\varepsilon]]\to\mathrm{Hom}\big(Y,X\big)\,,$$

which satisfies $f_i\circ\widehat{\sigma}=0$ for all i, and whose first order part agrees with σ. Artin's result therefore applies.

3. Strategy for the proof of Theorem 1.5

3.A Introduction

Before giving a complete proof of Theorem 1.5 in Section 6 below, we first outline the main strategy of the proof and recall a few elementary facts. We hope that the explanations given below will help to motivate the preparatory Sections 4 and 5 where we gather several technical results used in the proof.

We will constantly use a number of elementary facts concerning vector fields on manifolds, their associated ordinary differential equations, flow maps and local actions of 1-parameter groups. Since all relevant results hold without change in the holomorphic as well as in the C^∞ category, we have chosen to use [War83] as our main reference, for the reader's convenience. A thorough introduction to vector fields and their flows on possibly singular complex spaces is found in [Kau65].

3.B Outline of the proof

To start the outline, consider the setup of Theorem 1.5 in the simple case where $f: Y \to X$ is a closed immersion and where both X and Y are compact. Viewing Y as a subspace of X, let

$$\sigma \in \operatorname{Image}\Big(H^0\big(Y, \mathcal{F}|_Y\big) \to H^0\big(Y, T_X|_Y\big)\Big)$$

be a first order infinitesimal deformation of f along $\mathcal{F}$.

If σ is the restriction of a global vector field $D \in H^0\big(X, \mathcal{F}\big)$, we can integrate the vector field D globally on X, obtaining a holomorphic action of a 1-parameter group, say

$$\phi: \Delta \times X \to X,$$

such that for each point $x \in X$, the arc $\gamma_x: \Delta \to X$, $t \mapsto \phi(t,x)$ is a solution to the initial value problem associated with the ordinary differential equation described by D. In down-to-earth terms, the germ of γ_x is the unique solution to the problem of finding a germ of an arc $\gamma: \Delta \to X$ that satisfies the two following requirements,

$$\gamma(0) = x \qquad \text{and} \tag{3.0.1}$$

$$\gamma'(t) = D\big(\gamma(t)\big) \qquad \text{for all } t \in \Delta. \tag{3.0.2}$$

Notation 3.1. We call γ_x the *integral curve of D through x*.

Viewing ϕ as a deformation of f, this gives a proof of Theorem 1.5 in case σ comes from a global vector field. For this, observe that requirement (1.5.1) of Theorem 1.5 immediately follows from (3.0.2) above.

If σ is the restriction of a vector field $D \in H^0(U, \mathcal{F})$ that is defined only on an open neighborhood U of Y, but perhaps not on all of X, essentially the same strategy applies. In this setup, there exists a *local* action, cf. [War83, Thm 1.48]. More precisely, there exists an open, relatively compact neighborhood V of Y with $Y \subseteq V \Subset U$, there exists a disk Δ and a map

$$\phi \colon \Delta \times V \to U,$$

such that the arcs $t \mapsto \phi(t,x)$ are again solutions to the initial value problems (3.0.2). As before ϕ gives a deformation of f that solves the problem.

In general, however, σ is not the restriction of a vector field that lives on a neighborhood of Y, and extensions of σ to open subsets of X exist only locally, cf. [War83, Rem. 1.52]. More precisely, there exist finitely many open sets U_i that are open in X, cover Y and admit vector fields $D_i \in H^0(U_i, T_X)$ whose restrictions $D_i|_{Y\cap U_i}$ equal $\sigma|_{Y\cap U_i}$. As before, we find relatively compact open subsets $V_i \Subset U_i$ that still cover Y, and local action morphisms

$$\phi_i \colon \Delta_i \times V_i \to U_i,$$

again with the property that if x is a point in V_i, we obtain an arc $\gamma_{x,i} \colon \Delta \to X$ that solves the initial value problem for D_i, as in (3.0.1) and (3.0.2) above. However, if $i \neq j$ are any two indices, the local action morphisms will generally not agree on the overlap $V_i \cap V_j$, and if x is in $V_i \cap V_j \cap Y$, the arcs $\gamma_{x,i}$ and $\gamma_{x,j}$ will likewise not agree.

There are a few things we can say about $\gamma_{x,i}$ and $\gamma_{x,j}$, though. Since

$$D_i|_{V_i\cap V_j\cap Y} = D_j|_{V_i\cap V_j\cap Y} = \sigma|_{V_i\cap V_j\cap Y}$$

and since $\gamma_{x,i}$ and $\gamma_{x,j}$ satisfy (3.0.2), it is clear that for any point $x \in V_i \cap V_j \cap Y$, the arcs $\gamma_{x,i}$ and $\gamma_{x,j}$ agree to first order, though perhaps not to second order. In other words, the ϕ_i induce sections [1])

$$\tau^2_{D_i} \colon V_i \cap Y \to \mathrm{Jet}^2(X)|_{V_i\cap Y} \qquad \text{and} \qquad \tau^2_{D_j} \colon V_j \cap Y \to \mathrm{Jet}^2(X)|_{V_j\cap Y}$$

whose first-order parts $\pi_{2,1}(\tau_\bullet) \colon V_\bullet \cap Y \to \mathrm{Jet}^1(X)|_{V_\bullet\cap Y}$ agree on the overlap $V_i \cap V_j \cap Y$. We have seen in Section 2.C that the difference $\tau^2_{D_j} - \tau^2_{D_i}$ can be

[1]) Since the γ_x are holomorphic for each x, the ϕ_i give sections in $\mathrm{Jet}^n(X)|_{V_\bullet\cap Y}$, for any number n. For the purposes of this outline, we concentrate on the case $n = 2$.

expressed as a section of $T_X|_{V_i \cap V_j \cap Y}$, and we will see in Theorem 4.3 below that this difference is expressed in terms of the Lie bracket of the vector fields $D_\bullet$, as follows:

$$\tau^2_{D_j}|_{V_i \cap V_j \cap Y} - \tau^2_{D_i}|_{V_i \cap V_j \cap Y} = \big[D_i, D_j\big]|_{V_i \cap V_j \cap Y}.$$

This will allow us to describe the Čech cocycles associated with the problem of lifting the infinitesimal deformation σ from first to second order in terms of Lie brackets. An argument similar to the proof of Lemma 2.14 will then allow us to adjust the vector fields D_i, in such a way that the associated local group actions give a well-defined lifting of σ to second order, globally along Y. An iterated application of this method will give liftings to arbitrary order.

4. Jets associated with vector fields

If D_1 and D_2 are two vector fields on X and $x \in X$ is a point, the integral curves γ_i of D_i through x do generally not agree. If γ_1 and γ_2 agree to n-th order, we have seen that the difference between the $(n+1)$-th order parts of the γ_i can be expressed as an element $\vec{v} \in T_X|_x$. In this section, we aim to express $\vec{v}$ purely in terms of the vector fields D_i and their Lie brackets. Before formulating the result in Theorem 4.3 below, we need to introduce some notation.

Definition 4.1 (Jets associated with a vector field). Let $U \subseteq X$ be an open set, and let $D \in H^0\big(U, T_X\big)$ be a vector field. Given any number $n \in \mathbf{N}$, let $\tau^n_D\colon U \to \mathrm{Jet}^n(X)$ be the section in the n-th jet bundle induced by the local action of the vector field.

In other words, if $x \in U$ is any point, and $\gamma_x\colon \Delta \to X$ the unique curve germ that satisfies (3.0.1) and (3.0.2), then $\tau^n_D(x)$ is exactly the n-th order jet associated with γ_x.

Definition 4.2 (Iterated Lie brackets). Let $U \subseteq X$ be an open set, and let $D_1, D_2 \in H^0\big(U, T_X\big)$ be two vector fields. For any integer $n \geq 2$, we recursively define a vector field, called the *n-th iterated Lie bracket of D_1 and D_2*, as follows:

$$[D_1, D_2]^{(2)} := [D_1, D_2] \qquad \text{and} \qquad [D_1, D_2]^{(n)} := \big[D_1, [D_1, D_2]^{(n-1)}\big].$$

THEOREM 4.3. *Let $U \subseteq X$ be an open set, and let $D_1, D_2 \in H^0\big(U, T_X\big)$ be two vector fields. If $x \in U$ is any point and n any integer such that the n-th order jets associated with D_1 and D_2 agree at x, i.e. $\tau^n_{D_1}(x) = \tau^n_{D_2}(x)$, then the tangent vector that describes the difference between the $(n+1)$-th order jets is expressed in terms of iterated Lie brackets as follows:*

$$\tau^{n+1}_{D_2}(x) - \tau^{n+1}_{D_1}(x) = [D_1, D_2]^{(n+1)}|_x. \tag{4.3.1}$$

Proof of Theorem 4.3 *for* $n = 1$. Choose a coordinate neighborhood U of x and let $\gamma_i \colon \Delta \to X$ be the germs of the integral curves of D_i through x for $i \in \{1, 2\}$. By Remark 2.11 and Fact 2.8, the difference between the second order parts of the γ_i is then expressed in U-coordinates as the difference of the second derivatives,

$$\vec{v} := \tau^2_{D_2}(x) - \tau^2_{D_1}(x) = \gamma_2''(0) - \gamma_1''(0) \in T_X|_x. \tag{4.3.2}$$

We aim to express the right hand side of (4.3.2) in terms of the vector fields D_i. For that, it is convenient to recall that to give a vector field D on U, it is equivalent to give a derivation $\mathcal{O}_X|_U \to \mathcal{O}_X|_U$, written as $f \mapsto Df$. Likewise, to give a tangent vector at x, it is equivalent to give a derivation $\mathcal{O}_{X,x} \to \mathbf{C}$, where $\mathcal{O}_{X,x}$ denotes the stalk of $\mathcal{O}_X$ at x. For a given tangent vector $\vec{w} \in T_X|_x$, the derivation is $f \mapsto f'(x) \cdot \vec{w}$, where f' is the derivative of f in U-coordinates, and the dot is matrix-vector multiplication. The derivations commute with restriction, so that $(Df)(x) = f'(x) \cdot D|_x$ for all f.

Now, if $f \in \mathcal{O}_{X,x}$ is any germ of a function, taking the second derivative of $f \circ \gamma_i$ yields

$$f'(x) \cdot \vec{v} = f'(x) \cdot \big(\gamma_2''(0) - \gamma_1''(0)\big) = (f \circ \gamma_2 - f \circ \gamma_1)''(0). \tag{4.3.3}$$

In order to relate the right hand side of (4.3.3) to the vector fields D_i, recall Equation (3.0.2), which asserts that for any function g, we have $(g \circ \gamma_i)' = (D_i g) \circ \gamma_i$. Applying this to f and $D_i f$, we obtain the following expression for the second derivatives of $f \circ \gamma_i$,

$$(f \circ \gamma_i)'' = \big((D_i f) \circ \gamma_i\big)' = \big(D_i(D_i f)\big) \circ \gamma_i = (D_i^2 f) \circ \gamma_i. \tag{4.3.4}$$

Substituting (4.3.4) into (4.3.3) we find that the equality

$$f'(x) \cdot \vec{v} = \big((D_2^2 - D_1^2) f\big)(x)$$

holds for all $f \in \mathcal{O}_{X,x}$, and therefore expresses $\vec{v}$ in terms of the vector fields D_i. To prove (4.3.1), it is therefore sufficient to show that

$$\big((D_2^2 - D_1^2) f\big)(x) = \big([D_1, D_2] f\big)(x) = \big((D_1 D_2 - D_2 D_1) f\big)(x) \tag{4.3.5}$$

holds true for all $f \in \mathcal{O}_{X,x}$. We show a stronger statement: for all f we have

$$(4.3.6) \qquad (D_1^2 f)(x) = (D_2 D_1 f)(x) \quad \text{and} \quad (D_2^2 f)(x) = (D_1 D_2 f)(x).$$

To prove (4.3.6), note that the equality $D_1|_x = D_2|_x$ implies that $(D_1 g)(x) = (D_2 g)(x)$ for every $g \in \mathcal{O}_{X,x}$. An application to $g = D_1 f$ and $g = D_2 f$, respectively, gives the two equalities in (4.3.6).

Sketch of proof of Theorem 4.3 *for arbitrary* n. The line of argumentation used to show Theorem 4.3 in case $n = 1$ also works for arbitrary n. As a first step, one shows that the difference vector $\vec{v} := \tau_{D_2}^{n+1}(x) - \tau_{D_1}^{n+1}(x) = \gamma_2^{(n+1)}(0) - \gamma_1^{(n+1)}(0) \in T_X|_x$ is determined by that fact that it satisfies the equation

$$(4.3.7) \qquad f'(x) \cdot \vec{v} = \big((D_2^{n+1} - D_1^{n+1}) f\big)(x)$$

for all $f \in \mathcal{O}_{X,x}$. Once this is established, it remains to show that

$$(4.3.8) \qquad \big((D_2^{n+1} - D_1^{n+1}) f\big)(x) = \big([D_1, D_2]^{(n+1)} f\big)(x),$$

again for all $f \in \mathcal{O}_{X,x}$. Equations (4.3.7) and (4.3.8) can be shown by induction on n, using elementary but tedious computations in local coordinates. We refer to [Loh08, Satz 1.4] for details.

5. Frobenius theorems and deformations along subsheaves

In Theorem 1.5, we aim to deform the morphism f along the sheaf $\mathcal{F}$. For that, we aim to define an analytic subspace $\mathrm{Hom}_{\mathcal{F}}(Y, X) \subseteq \mathrm{Hom}(Y, X)$ which parametrizes such deformations. If $\mathcal{T}$ is a regular foliation, the space $\mathrm{Hom}_{\mathcal{F}}(Y, X)$ can be defined as a relative analytic Douady space of morphisms, using the classical Frobenius Theorem which asserts that $\mathcal{T}$ is the foliation associated with a morphism, at least locally.

THEOREM 5.1 (Frobenius Theorem, cf. [War83, Thm 1.60]). *Let Z be a complex manifold and $\mathcal{G} \subset T_Z$ a regular foliation, i.e., a vector subbundle of T_Z which is closed under the Lie bracket. If $z \in Z$ is any point, then there exists an analytic neighborhood $U = U(z) \subset Z$ which has a product structure, $U = A \times B$, such that $\mathcal{G} = \pi_A^*(T_A)$, where $\pi_A\colon A \times B \to A$ is the projection to the first factor.*

After introducing some notation and after proving the auxiliary Proposition 5.3, we give a generalization of the Frobenius Theorem that works for arbitrary Lie-closed sheaves. While this result, formulated in Corollary 5.4, is probably known to experts, we include a full proof, for lack of an adequate reference. We will use this version of the Frobenius Theorem to define the space $\mathrm{Hom}_{\mathcal{F}}(Y, X)$ in Corollary 5.6 and to prove some of its universal properties.

Throughout the present section, we maintain the notation of Theorem 1.5 where X is a complex manifold and $\mathcal{F} \subseteq T_X$ a sheaf which is closed under the Lie bracket.

NOTATION 5.2 (Stratification of X). It follows immediately from semicontinuity of rank that for any integer r, the subset

$$X_r := \{x \in X \mid \mathrm{rank}(\mathcal{F}|_x \to T_X|_x) = r\} \subseteq X$$

is a locally closed analytic subspace of X. We consider the natural sequence of closed analytic subspaces of X_r,

$$X_r = X_r^0 \supseteq X_r^1 \supseteq \cdots \supseteq X_r^{m_r} \supseteq X_r^{m_r+1} = \varnothing,$$

where X_r^{i+1} is defined inductively as the *singular locus* of X_r^i. We obtain a decomposition of X into finitely many disjoint, smooth and locally closed analytic subspaces,

$$X = \dot{\bigcup_{r,s}} Z_r^s \qquad \text{with} \qquad Z_r^s := X_r^s \setminus X_r^{s+1}.$$

PROPOSITION 5.3. *Let r be any number such that $X_r \neq \varnothing$, let $x \in X_r$ be any point and $D \in H^0(U, \mathcal{F})$ a vector field, defined in a neighborhood $U = U(x) \subseteq X$ of x. If $\gamma_x\colon \Delta \to X$ is the integral curve of D through x, as defined in Notation* 3.1, *then $\gamma_x(t) \in X_r$ for all $t \in \Delta$.*

Proof. Let $q \leq r$ be the least integer such that the set $\Delta_q := \gamma_x^{-1}(X_q)$ is not empty. By semicontinuity, $\Delta_q \subseteq \Delta$ is a closed analytic subset, and to prove Proposition 5.3, it suffices to show that Δ_q is also open. Using the fundamental property that $\gamma_{\gamma(t_0)}(t) = \gamma_x(t + t_0)$ for all $t_0 \in \Delta_q$ and all sufficiently small numbers t, we can assume without loss of generality that $0 \in \Delta_q$ and $r = q$. For the same reason, it suffices to show that Δ_q contains a neighborhood Δ' of $0 \in \Delta$.

To this end, we will show that near $0 \in \Delta$, the local group action induced by D yields an injective linear map from $\mathrm{Image}(\mathcal{F}|_{\gamma_x(t)} \to T_X|_{\gamma_x(t)})$ to a q-dimensional vector space, for every sufficiently small number t. Shrinking U, if necessary, we can assume without loss of generality that the sheaf $\mathcal{F}|_U$ is generated by vector fields $D_1, \ldots, D_s \in H^0\big(U, \mathcal{F}\big)$. The vector field D induces a local group action $\phi\colon \Delta' \times V \to U$, where $V \subseteq U$ and $\Delta' \subseteq \Delta$ are suitably small open neighborhoods of x and 0, respectively.

To prove Proposition 5.3, we need to show that $\Delta' \subseteq \Delta_q$. For this, pick any element $t \in \Delta'$ and set $y := \gamma_x(t)$. We consider the vector spaces

$$W_x := \langle D_1(x), \ldots, D_s(x)\rangle \subseteq T_X|_x \quad \text{and} \quad W_y := \langle D_1(y), \ldots, D_s(y)\rangle \subseteq T_X|_y .$$

Since $W_x = \mathrm{Image}(\mathcal{F}|_x \to T_X|_x)$, the dimension of W_x equals $r = q$, and since q is chosen minimal, Proposition 5.3 is shown once we prove that $\dim W_y \leq q$. In order to relate the spaces W_x and W_y we consider the open immersion $\phi_t\colon V \to U$, $\phi_t(v) := \phi(t, v)$, whose pull-back morphism yields an isomorphism of vector spaces $\phi_t^*\colon T_X|_y \to T_X|_x$.

To understand the morphism ϕ_t^* better, let $D_i(y)$ be any generator of W_y, and define the map

$$\begin{array}{cccc} \Gamma\colon & \Delta' & \longrightarrow & T_X|_x \\ & t' & \longmapsto & (\phi_{t'}^* \circ D_i)\big(\phi_{t'}(x)\big) \end{array}$$

and notice that $\Gamma(0) = D_i(x) \in W_x$ and $\Gamma(t) = \phi_t^*\big(D_i(y)\big)$. Since ϕ_t^* is injective, it remains to prove that $\Gamma(t)$ is an element of W_x. Recall from [War83, Def. 2.24, Prop. 2.25] that Γ is analytic and that its derivative is

$$\Gamma'(t') = (\phi_{t'}^* \circ [D, D_i])\big(\phi_{t'}(x)\big).$$

In particular, we have that $\Gamma'(0) = [D, D_i](x)$ is an element of W_x. It follows by induction that the higher-orderderivatives are given by

$$\Gamma^{(n)}(t') = (\phi_{t'}^* \circ [D, D_i]^{(n)})\big(\phi_{t'}(x)\big).$$

In particular, we have that $\Gamma^{(n)}(0) = [D, D_i]^{(n)}(x)$ is an element of W_x, for all numbers n. Expanding Γ in a Taylor series, it follows that $\Gamma(t')$ is an element of W_x, for all $t' \in \Delta'$.

In summary, we see that the isomorphism $\phi_t^*\colon T_X|_y \to T_X|_x$ maps each generator $D_i(y)$ of W_y to W_x. As a consequence, we obtain $\dim W_y \leq \dim W_x = q$, as claimed. This ends the proof of Proposition 5.3.

COROLLARY 5.4 (Frobenius Theorem for $\mathcal{F}$). *If r, s are any two integers such that $Z := Z_r^s$ is not empty, then*

(5.4.1) *the image of $\mathcal{T}$ along Z is contained in the tangent bundle of Z, i.e.,*

$$\mathcal{T}_Z := \operatorname{Image}(\mathcal{T}|_Z \to T_X|_Z) \subseteq T_Z,$$

(5.4.2) *the sheaf $\mathcal{F}_Z \subseteq T_Z$ is a regular foliation, and*

(5.4.3) *every point $z \in Z$ admits an open neighborhood $U = U(z) \subseteq Z$ with a product structure, $U = A \times B$ such that $\mathcal{F}_Z \cong \pi_A^*(T_A)$, where $\pi_A \colon A \times B \to A$ is the projection to the first factor.*

Proof. Let $U \subseteq X$ be any open subset of X, and let $D \in H^0(U, \mathcal{F})$ be any vector field, with an associated local group action $\phi \colon \Delta \times V \to X$, where Δ is again a sufficiently small disk and $V \subseteq U$ a suitable open subset that contains x. By Proposition 5.3, we know that for any point $x' \in V$ and any $t \in \Delta$, we have $\phi(t, x') \in X_r$ if and only if $x' \in X_r$. In fact, more is true: since the morphisms $\phi(t, \cdot) \colon V \to X$ are open immersions, they must stabilize the singular set of X_r. Eventually, it follows that for any number s, we have $\phi(t, x') \in X_r^s$ if and only if $x' \in X_r^s$. Since

$$D|_x = \left(\tfrac{\partial}{\partial t}\phi\right)(0, x) = \left(\tfrac{\partial}{\partial t}\gamma_x\right)(0),$$

this implies Claim (5.4.1).

By definition of X_r, it is clear that $\mathcal{F}_Z$ is a vector subbundle of T_Z. The assertion that $\mathcal{F}_Z$ is closed under the Lie bracket of T_Z follows from Claim (5.4.1) and a standard computation, cf. [War83, Prop. 1.55], giving Claim (5.4.2). Claim (5.4.3) follows when one applies the classical Frobenius Theorem 5.1 to $\mathcal{F}_Z \subseteq T_Z$.

Using Corollary 5.4, we can now define the analytic space $\operatorname{Hom}_{\mathcal{F}}(Y, X)$ which parametrizes deformations along $\mathcal{F}$. The following notation is useful for the description of its universal properties.

DEFINITION 5.5 (Infinitesimal deformations that are pointwise induced by a subsheaf). Let $\sigma_n \colon Y \to f^* \operatorname{Jet}^n(X)$ be an n-th order infinitesimal deformation of the morphism f. We say that σ_n is *pointwise induced by vector fields in* $\mathcal{F}$, if for any point $y \in Y$ there is a neighborhood $U \subseteq X$ of $f(y)$ and a vector field $D \in H^0(U, \mathcal{T})$ such that $\sigma_n(y) = \tau_D^n(f(y))$, where $\tau_D^n \colon U \to \operatorname{Jet}^n(X)$ is the section in the n-th jet bundle described in Definition 4.1.

COROLLARY 5.6 (Existence of a parameter space for deformations along a subsheaf). *There exists a locally closed analytic subspace* $\mathrm{Hom}_{\mathcal{F}}(Y, X) \subseteq \mathrm{Hom}(Y, X)$ *which contains the morphism* f *and has the following universal properties.*

(5.6.1) *If* σ_n *is an* n*-th order infinitesimal deformation of the morphism* f *which is pointwise induced by vector fields in* $\mathcal{F}$, *then the associated morphism* $\mathrm{Spec}\,\mathbf{C}[\varepsilon]/(\varepsilon^{n+1}) \to \mathrm{Hom}(Y, X)$ *factors via* $\mathrm{Hom}_{\mathcal{F}}(Y, X)$.

(5.6.2) *If* $\gamma\colon \Delta \to \mathrm{Hom}_{\mathcal{F}}(Y, X)$ *is any arc with* $\gamma(0) = f$, *and if* $F\colon \Delta \times Y \to X$ *is the associated deformation, then* F *is a deformation along* $\mathcal{F}$, *in the sense of Notation* 1.6.

Proof. We begin with the construction of the space $\mathrm{Hom}_{\mathcal{F}}(Y, X)$. Choose integers r, s with $f(Y) \cap Z_r^s \neq \varnothing$, an irreducible component $Y' \subseteq f^{-1}(Z_r^s)$, a general point $y_0 \in Y'$ and a neighborhood $V = V(f(y_0)) \subseteq X$, with a decomposition $V \cap Z_r^s = A \times B$ as in Corollary 5.4. Let $U = U(y_0) \Subset Y' \cap f^{-1}(V)$ be a relatively compact neighborhood. By relative compactness of U, there exists an analytically open neighborhood $H^1_{r,s,Y'} \subseteq \mathrm{Hom}(Y, X)$ of $f \in \mathrm{Hom}(Y, X)$ such that $g(y) \in V$ for all points $y \in U$ and all morphisms $g \in H^1_{r,s,Y'}$. The set

$$H^2_{r,s,Y'} := \bigcap_{y \in U} \{ g \in H^1_{r,s,Y'} \mid g(y) \in Z_r^s \} \subseteq H^1_{r,s,Y'} \tag{5.6.3}$$

is then the intersection of finitely or infinitely many analytic subspaces, and therefore, by the analytic version of Hilbert's Basissatz [KK83, Prop. 23.1], an analytic subspace itself. We remark that neither $H^2_{r,s,Y'}$ nor any of the spaces on the right hand side of (5.6.3) are necessarily reduced.

Identifying $V \cap Z_r^s \cong A \times B$, with projection $\pi_B\colon A \times B \to B$, we can then consider the following analytic subspace of $H^2_{r,s,Y'}$:

$$H_{r,s,Y'} := \bigcap_{y \in U} \{ g \in H^2_{r,s,Y'} \mid (\pi_B \circ g)(y) = (\pi_B \circ f)(y) \} \subseteq H^2_{r,s,Y'}. \tag{5.6.4}$$

In order to define the subspace $\mathrm{Hom}_{\mathcal{F}}(Y, X) \subseteq \mathrm{Hom}(Y, X)$, repeat this construction for each of the finitely many numbers r and s, and for each of the finitely many components $Y' \subseteq f^{-1}(Z_r^s)$. Finally, let $\mathrm{Hom}_{\mathcal{F}}(Y, X)$ be the connected component of the intersection which contains f,

$$\mathrm{Hom}_{\mathcal{F}}(Y, X) \subseteq \bigcap_{r,s,Y'} H_{r,s,Y'} \subseteq \underbrace{\bigcap_{r,s,Y'} H^1_{r,s,Y'}}_{\text{open in } \mathrm{Hom}(Y,X)} \subseteq \mathrm{Hom}(Y, X).$$

It remains to show that the Universal Properties (5.6.1) and (5.6.2) hold.

For Property (5.6.1), assume that an n-th order deformation σ_n is given as in (5.6.1). Given any two integers r, s and any connected component $Y' \subseteq f^{-1}(Z_r^s)$, let $V \subseteq X$ and $U \Subset Y' \cap f^{-1}(V)$ be the sets considered above in the construction of $H_{r,s,Y'}$, with decomposition $V \cap Z_r^s \cong A \times B$. Now, if $y \in U$ is any point and D the associated vector field near $f(y)$, with integral curve $\gamma_{f(y)}\colon \Delta \to X$, it is clear from Corollary 5.4 that $\gamma_{f(y)}(t) \in Z_r^s$, for all t. In particular, the associated morphism

$$\sigma_n\colon \operatorname{Spec} \mathbf{C}[\varepsilon]/(\varepsilon^{n+1}) \to \operatorname{Hom}\bigl(Y, X\bigr) \tag{5.6.5}$$

factors via $H^2_{r,s,Y'}$. In a similar vein, it follows from Corollary 5.4 that $\pi_B \circ \gamma_{f(y)}(t) \equiv \pi_B\bigl(f(y)\bigr)$ for all $t \in \Delta$. In particular, viewing σ_n as a map $\sigma_n\colon \operatorname{Spec} \mathbf{C}[\varepsilon]/(\varepsilon^{n+1}) \times Y \to X$, we have

$$\pi_B \circ \bigl(\sigma_n|_{\operatorname{Spec} \mathbf{C}[\varepsilon]/(\varepsilon^{n+1}) \times \{y\}}\bigr) \equiv \pi_B\bigl(f(y)\bigr),$$

so that the morphism (5.6.5) actually factors via $H_{r,s,Y'}$. Since this is true for all r, s and Y', the morphism (5.6.5) factors via $\operatorname{Hom}_{\mathcal{F}}\bigl(Y, X\bigr)$, as claimed. This ends the proof of Property (5.6.1).

To prove Property (5.6.2), let γ be any arc that satisfies the conditions of (5.6.2) and let F be the associated deformation. For $t \in \Delta$, let

$$\sigma_{F,t} \in H^0\bigl(Y, (F_t)^*(T_X)\bigr)$$

be the velocity vector field, as introduced in Notation 1.3 on page 289. We aim to show that the $\sigma_{F,t}$ are really sections in $(F_t)^*(\mathcal{F})$. Again, if any two integers r, s and any connected component $Y' \subseteq f^{-1}(Z_r^s)$ are given, it is clear from (5.6.3) and (5.6.4) that

$$\sigma_{F,t}|_U \in H^0\bigl(U, (F_t)^*(\mathcal{F}_{Z_r^s})\bigr),$$

where $\mathcal{F}_{Z_r^s}$ is the sheaf introduced in Corollary 5.4. Since U is analytically open in the irreducible space $Y' \subseteq f^{-1}(Z_r^s)$ and since we have seen in Corollary 5.4 that $\mathcal{F}_{Z_r^s}$ is a vector bundle, it follows immediately from the identity principle that

$$\sigma_{F,t}|_{Y'} \in H^0\bigl(Y', (F_t)^*(\mathcal{F}_{Z_r^s})\bigr).$$

Since this holds for all numbers r and s, and all irreducible components $Y' \subseteq f^{-1}(Z_r^s)$, Property (5.6.2) follows.

6. Proof of Theorem 1.5

6.A Setup of notation, overview of the proof

We end this paper with the proof of Theorem 1.5. Throughout the present Section 6, we maintain the assumptions and the notation of the theorem. In particular, we assume that we are given a morphism $f\colon Y \to X$ of complex manifolds, with Y compact, an involutive subsheaf $\mathcal{F} \subseteq T_X$ and a first order infinitesimal deformation of f, denoted $\sigma \in H^0\big(Y, \mathcal{F}_Y\big)$, where $\mathcal{F}_Y \subseteq f^*(T_X)$ is the image of $f^*(\mathcal{F})$ under the pull-back of the inclusion map. We also assume that $H^1\big(Y, \mathcal{F}_Y\big) = \{0\}$.

The proof is given in three steps. Replacing the target manifold X with the product $Y \times X$, and the morphism f with the natural graph map, we first show that it suffices to prove Theorem 1.5 in the case where f is a closed immersion. In Step 2, we construct a setting where the tangent vectors $\sigma(y) \in T_X|_y$ and the vector spaces $T_Y|_y \subseteq T_X|_y$ are transversal at all points $y \in Y$. A third step will then complete the proof.

6.B Step 1: Reduction to the case of a closed immersion

In Section 3.B we have discussed the situation where f is a closed immersion, and where the infinitesimal deformation σ was locally given by restrictions of vector fields that live on open subsets of X. In order to reduce to this simpler situation, we will show that to give a deformation of f, it is equivalent to give a relative deformation of the graph morphism,

$$\iota\colon Y \to Y \times X, \quad \text{where} \quad \iota(y) = \big(y, f(y)\big),$$

which is a closed immersion that identifies the domain Y with the graph of f. We will then aim to construct an involutive subsheaf $\mathcal{G} \subseteq T_{Y\times X}$ that comes from $\mathcal{F}$, and an infinitesimal deformation of the graph morphism ι along the sheaf $\mathcal{G}$ that is related to σ.

For this, recall that the tangent bundle of the product is a direct sum $T_{Y\times X} = \pi_Y^*(T_Y) \oplus \pi_X^*(T_X)$, where the $\pi_\bullet$ are the natural projections, and set $\mathcal{G} := \{0\} \oplus \pi_X^*(\mathcal{F}) \subseteq T_{Y\times X}$. Since $\mathcal{G}$ is generated by vector fields that are π_X-related to vector fields in $\mathcal{F}$, it follows from [War83, Prop. 1.55] that $\mathcal{G}$ is closed under the Lie bracket. Finally, consider the first order infinitesimal deformation $\sigma_\iota\colon Y \to \iota^*(T_{Y\times X})$ of ι, given by $\sigma_\iota(y) := (0, \sigma(y))$.

The following lemmas are then immediate from the construction.

Lemma 6.1. *The infinitesimal deformation σ_ι is contained in the subspace* $H^0\big(Y, \operatorname{Image}(\iota^*(\mathcal{G}) \to \iota^*(T_{X\times Y}))\big) \subseteq H^0\big(Y, \iota^*(T_{X\times Y})\big)$.

LEMMA 6.2. *There exist natural isomorphisms* $\iota^*(\mathcal{G}) \cong f^*(\mathcal{F})$ *and*

$$\mathrm{Image}\big(\iota^*(\mathcal{G}) \to \iota^*(T_{X\times Y})\big) \cong \mathrm{Image}\big(f^*(\mathcal{F}) \to f^*(T_X)\big) =: \mathcal{F}_Y .$$

In particular, we have

$$H^1\big(Y,\, \mathrm{Image}(\iota^*(\mathcal{G}) \to \iota^*(T_{X\times Y}))\big) = H^1\big(Y,\, \mathcal{F}_Y\big) = \{0\}.$$

LEMMA 6.3. *If* $F_\iota\colon \Delta\times Y \to Y\times X$ *is a deformation of the graph morphism* ι *along* $\mathcal{G}$, *then* $F := \pi_X \circ F_\iota\colon \Delta\times Y \to X$ *is a deformation of* f *along* $\mathcal{F}$. *If* F_ι *is a lifting of* σ_ι, *then* F *is a lifting of* σ.

In summary, Lemmas 6.1–6.3 show that all assumptions made in Theorem 1.5 also hold for the morphism ι, and that it suffices to find a lifting of σ_ι along $\mathcal{G}$. Without loss of generality, we can therefore maintain the following assumption throughout the rest of the proof.

ASSUMPTION 6.4. *The morphism* $f\colon Y \to X$ *is a closed immersion.*

6.C STEP 2: TIME-DEPENDENT VECTOR FIELDS

The explicit computations of Čech cocycles that we will use in Step 3 of this proof become rather complicated if the infinitesimal deformation σ has zeros or if its associated tangent vectors are not transversal to $f(Y) \subseteq X$. As in Section 6.B, we avoid this problem by enlarging X.

CONSTRUCTION 6.5. Set $Z := X \times \mathbf{C}$, with projections $\pi_X\colon Z \to X$ and $\pi_{\mathbf{C}}\colon Z \to \mathbf{C}$. Throughout the remainder of the proof, the coordinate on $\mathbf{C}$ will be denoted by t and referred to as "time". Using that the tangent bundle of Z decomposes as a direct sum, we consider the sheaf

$$\mathcal{G} := \pi_X^*(\mathcal{F}) \oplus \pi_{\mathbf{C}}^*(T_{\mathbf{C}}) \subseteq \pi_X^*(T_X) \oplus \pi_{\mathbf{C}}^*(T_{\mathbf{C}}) = T_Z ,$$

the inclusion map

$$g\colon Y \to Z, \quad y \mapsto \big(f(y), 0\big)$$

and the infinitesimal deformation

$$\eta \in H^0\big(Y,\, g^*(T_Z)\big), \quad \eta := \sigma + \frac{d}{dt}.$$

As in Section 6.B, the following is immediate from the construction:

LEMMA 6.6. *The sheaf $\mathcal{G}$ is closed under Lie bracket. If $G\colon \Delta \times Y \to Z$ is a deformation of the morphism g along $\mathcal{G}$, then $F := \pi_X \circ G\colon \Delta \times Y \to X$ is a deformation of f along $\mathcal{F}$. If the deformation G is a lifting of η, then F is a lifting of σ.*

WARNING 6.7. If $\mathcal{G}_Y \subseteq g^*(T_Z)$ denotes the image of $g^*(\mathcal{G})$ under the pull-back of the inclusion map, then $\mathcal{G}_Y = \mathcal{F}_Y \oplus \mathcal{O}_Y$. It is therefore generally wrong that $H^1(Y, \mathcal{G}_Y) = \{0\}$, and the assumptions of Theorem 1.5 will generally not hold for the morphism g. Rather than using cohomological vanishing for $\mathcal{G}_Y$, the arguments given in Step 3 will therefore only use cohomological vanishing of $\mathcal{F}_Y$ and the special form of η, in order to construct infinitesimal liftings of arbitrary order.

The following special types of vector fields on Z will play a role in the computations.

DEFINITION 6.8 (Time-dependent vector field). A vector field on Z is called a *time-dependent vector field in $\mathcal{F}$* if it is a section of the sheaf

$$\pi_X^*(\mathcal{F}) \oplus \{0\} \subseteq \pi_X^*(\mathcal{F}) \oplus \pi_{\mathbb{C}}^*(T_{\mathbb{C}}) \subseteq T_Z .$$

DEFINITION 6.9 (Vector field with constant flow in time). A vector field D on Z is called a *vector field in $\mathcal{G}$ with constant flow in time* if it is of the form

$$D = D' + \frac{d}{dt},$$

where D' is a time-dependent vector field in $\mathcal{F}$.

We remark that the first-order infinitesimal deformation η of Construction 6.5 is induced by a vector field with constant flow in time, in the sense of the following definition.

DEFINITION 6.10 (Infinitesimal deformations induced by vector fields). An n-th order infinitesimal deformation $\eta_n\colon Y \to g^* \operatorname{Jet}^n(Z) = \operatorname{Jet}^n(Z)|_Y$ of the closed immersion g is *induced by vector fields in $\mathcal{G}$ with constant flow in time* if for every point $y \in Y$ there are a neighborhood $U = U(g(y)) \subseteq Z$ and a vector field $D \in H^0(U, \mathcal{G})$ with constant flow in time, such that the restriction $\eta_n|_{U\cap Y}$ is given by the section $\tau_D^n|_{U\cap Y}$ discussed in Definition 4.1.

In Step 3 of the proof, we need to consider iterated Lie brackets of vector fields with constant flow in time. We end this section with an elementary observation, asserting that Lie brackets of time-dependent vector fields, or of vector fields with constant flow in time will always be time dependent.

LEMMA 6.11. *Let $U \subseteq Z$ be any open set and let D_1 and D_2 be any two time dependent vector fields in $\mathcal{F}$, defined on U. Then the following hold:*

(6.11.1) *The Lie bracket $[D_1, D_2]$ is a time-dependent vector field in $\mathcal{T}$.*

(6.11.2) *The Lie bracket $[\frac{d}{dt}, D_1]$ is a time-dependent vector field in $\mathcal{F}$.*

Proof. Assertion (6.11.(6.11.1)) follows from an elementary computation, cf. [War83, Prop. 1.55], when one observes that a vector field in $\mathcal{G}$ is a time-dependent vector field in $\mathcal{F}$ if and only if it is $\pi_{\mathbf{C}}$-related to the trivial vector field $0 \in H^0(\mathbf{C}, T_{\mathbf{C}})$. Observing that a vector field has constant flow in time if and only if it is $\pi_{\mathbf{C}}$-related to the vector field $\frac{d}{dt} \in H^0(\mathbf{C}, T_{\mathbf{C}})$, the same computation also gives (6.11.(6.11.2)).

COROLLARY 6.12. *Let $D_1 + \frac{d}{dt}$ and $D_2 + \frac{d}{dt}$ be any two vector fields in $\mathcal{G}$ with constant flow in time. If n is any integer, then the iterated Lie bracket $[D_1 + \frac{d}{dt}, D_2 + \frac{d}{dt}]^{(n)}$ is a time dependent vector field in $\mathcal{F}$.*

6.D STEP 3: END OF PROOF

The end of the proof of Theorem 1.5 is now very similar to the proof of Theorem 1.1. First, we prove an analogue of Lemma 2.14 that gives liftings of η to arbitrary order. These liftings will locally be induced by vector fields in $\mathcal{F}$ with constant flow in time. Finally, we apply Artin's result to construct the required deformation of f. The universal properties of the space $\mathrm{Hom}_{\mathcal{F}}(Y, X)$, as spelled out in Corollary 5.6, will then guarantee that this is in fact a deformation along the subsheaf $\mathcal{T}$.

LEMMA 6.13. *Let $\eta_n \colon Y \to g^* \mathrm{Jet}^n(Z)$ be an n-th order infinitesimal deformation of the closed immersion g that is induced by vector fields in $\mathcal{G}$ with constant flow in time. Then there exists a lifting $\eta_{n+1} \colon Y \to g^* \mathrm{Jet}^{n+1}(Z)$ of η_n that is likewise induced by vector fields in $\mathcal{G}$ with constant flow in time.*

Proof. As a first step, we construct liftings locally. Using the cohomological vanishing for $\mathcal{F}_Y$, we can then correct the local liftings, to ensure that they glue on overlaps. This will define a global lifting, which is then shown to be induced by vector fields in $\mathcal{G}$ with constant flow in time.

It follows from Definition 6.10 that there exists an acyclic covering of $g(Y) \subseteq Z$ by open subsets $(U_i)_{i\in I} \subseteq Z$ such that there are time-dependent vector fields $D_i \in H^0\big(U_i, \pi_X^*(\mathcal{F}) \oplus \{0\}\big)$ that satisfy $\eta_n|_{U_i\cap Y} = \tau^n_{D_i+\frac{d}{dt}}|_{U_i\cap Y}$. We consider the induced section of the $(n+1)$-th jet bundle,

$$\tau_i := \tau^{n+1}_{D_i+\frac{d}{dt}}\Big|_{U_i\cap Y} \colon U_i \cap Y \to \mathrm{Jet}^{n+1}(Z)\,.$$

Obviously, the τ_i are local liftings of η_n, but they do not necessarily glue on overlaps. However, it follows from Theorem 4.3 that for any pair of indices $i, j \in I$, the affine differences are given by iterated Lie brackets,

$$\begin{aligned}\nu_{i,j} &:= \tau_i\big|_{U_i\cap U_j\cap Y} - \tau_j\big|_{U_i\cap U_j\cap Y}\\ &= \underbrace{\left[D_i + \frac{d}{dt}\Big|_{U_i\cap U_j},\ D_j + \frac{d}{dt}\Big|_{U_i\cap U_j}\right]^{(n+1)}}_{=:A_{i,j}}\Bigg|_{U_i\cap U_j\cap Y}.\end{aligned}$$

Corollary 6.12 asserts that the iterated Lie brackets $A_{i,j}$ are time-dependent vector fields in $\mathcal{F}$. The differences $\nu_{i,j}$ therefore yield cohomology classes in $H^1\big(Y, \mathcal{F}_Y\big)$ which are zero by assumption. We can thus find sections $\lambda_i \in H^0\big(U_i \cap Y, \mathcal{F}_Y\big)$ such that $\lambda_i - \lambda_j = \nu_{i,j}$. As in the proof of Lemma 2.14, viewing the λ_i as sections in $H^0\big(U_i \cap Y, \mathcal{F}_Y \oplus \mathcal{O}_Y\big) = H^0\big(U_i \cap Y, \mathcal{G}_Y\big) \subseteq H^0\big(U_i \cap Y, g^*(T_Z)\big)$, the sections obtained by translation,

$$\tau_i - \lambda_i \colon U_i \cap Y \to \mathrm{Jet}^{n+1}(Z)|_{U_i\cap Y}\,,$$

glue on overlaps $U_i \cap U_j \cap Y$ and define a lifting to $(n+1)$-th order,

$$\eta_{n+1} \colon Y \to \mathrm{Jet}^{n+1}(Z)\,, \quad \text{with} \quad \eta_{n+1}|_{U_i\cap Y} = \tau_i - \lambda_i \text{ for all } i\,. \tag{6.13.1}$$

It remains to show that η_{n+1} is an infinitesimal deformation induced by vector fields in $\mathcal{G}$ with constant flow in time. To check the conditions of Definition 6.10, let $y \in Y$ be any point, and let $i \in I$ be any index with $g(y) \in U_i$. Then it suffices to construct a time-dependent vector field $D \in H^0\big(U_i, \pi_X^*(\mathcal{T})\big)$ such that $\eta_{n+1}|_{U_i\cap Y} = \tau^{n+1}_{D+\frac{d}{dt}}|_{U_i\cap Y}$.

To this end, consider the sections τ_i and λ_i defined above. Recall that τ_i is induced by the vector field $D_i + \frac{d}{dt}$. Since the covering of Z is acyclic, the section $\lambda_i \in H^0\big(U_i \cap Y, \mathcal{F}_Y\big)$ is given as the restriction of a vector field $E \in H^0\big(U_i, \pi_X^*(\mathcal{F})\big)$. Set $D := (D_i - \frac{t^n}{n!}E)$. With Theorem 4.3 at hand, it is then easy to compute the affine differences of $\tau^{n+1}_{D+\frac{d}{dt}}|_{U_i\cap Y}$ and $\tau^{n+1}_{D_i+\frac{d}{dt}}|_{U_i\cap Y}$ on U_i as

$$\tau^{n+1}_{D+\frac{d}{dt}}|_{U_i\cap Y} - \underbrace{\tau^{n+1}_{D_i+\frac{d}{dt}}|_{U_i\cap Y}}_{=\tau_i} = -E = -\lambda_i\,.$$

We obtain $\tau^{n+1}_{D+\frac{d}{dt}}|_{U_i\cap Y} = \tau_i - \lambda_i$ and Equation (6.13.1) then gives $\tau^{n+1}_{D+\frac{d}{dt}}|_{U_i\cap Y} = \eta_{n+1}|_{U_i\cap Y}$, as required.

Proof of Theorem 1.5*; end of proof.* Consider the analytic subset $\mathrm{Hom}_{\mathcal{G}}(Y,Z)$ of the Douady space $\mathrm{Hom}(Y,Z)$ constructed in Corollary 5.6 and the sequence of liftings η_1, η_2, ... of Lemma 6.13. By Proposition 2.13, we can view the η_i as morphisms $\operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^{i+1})\times Y\to Z$. Assertion (5.6.1) of Corollary 5.6 then implies that these morphisms factor via $\mathrm{Hom}_{\mathcal{G}}(Y,Z)$, for each i.

Arguing as in the proof of Theorem 1.1, only replacing $\mathrm{Hom}(Y,Z)$ by the analytic subspace $\mathrm{Hom}_{\mathcal{G}}(Y,Z)$, Artin's Theorem [Art68, Thm 1.2] guarantees the existence of a deformation G of g that factors via $\mathrm{Hom}_{\mathcal{G}}(Y,Z)$ and lifts the infinitesimal deformation η. Lemma 6.6 and Assertion (5.6.2) of Corollary 5.6 then implies that $F=\pi_X\circ G$ is in fact a deformation along $\mathcal{T}$ that lifts the infinitesimal deformation σ.

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(Reçu le 12 juin 2009; version révisée reçue le 29 mars 2010)

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