CASE 3: $X = \rho(0, 0, \mathbf{c})$, $\mathbf{c} \in \mathbf{R}^3$. The proof for this case is very similar to the previous case. Just interchange $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{b}$ and $\mathbf{c}$.

This completes the proof of invariance, and hence the proof of the proposition.

C.4 RELATION WITH OCTONIONS

Recall the basis e_i, f_i, U of Section 6 for V (imaginary split octonions) with its consequent multiplication table. Make the change of basis $e_i \mapsto -e_i$, keeping f_i, U as they were, thus changing the signs of some entries of the multiplication table. Use this new basis $E_i = -e_i, f_i$, U to identify V with $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}$ by setting $(\mathbf{x}, \mathbf{y}, z) = \sum x_i E_i + \sum y_i f_i + zU \in V$. Referring to the multiplication table we compute

$$\begin{aligned}(\mathbf{x}, \mathbf{y}, z)(\mathbf{x}', \mathbf{y}', z') = \big(&-\mathbf{y} \times \mathbf{y}' - z\mathbf{x}' + z'\mathbf{x}, \\ &\mathbf{x} \times \mathbf{x}' + z\mathbf{y}' - z'\mathbf{y},\ \tfrac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})\big) + 1\{zz' + \tfrac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{x}' \cdot \mathbf{y})\}.\end{aligned}$$

The last term is in the real part of the split octonions, and not in V. It follows from this formula that $(\mathbf{x}, \mathbf{y}, z)^2 = J$, of Cartan's claim (2) stated above. Multiplying out $(\mathbf{x}, \mathbf{y}, z)(d\mathbf{x}, d\mathbf{y}, dz)$ we find that

$$(\mathbf{x}, \mathbf{y}, z)(d\mathbf{x}, d\mathbf{y}, dz) = (\alpha, \beta, \tfrac{1}{2}(\gamma_1 - \gamma_2)) + 1\{\tfrac{1}{2}(\gamma_1 + \gamma_2)\},$$

where $\alpha, \beta, \gamma_1, \gamma_2$ are as in Cartan's claim (4). It follows that any element of $G_2 = \mathrm{Aut}(\widetilde{\mathbf{O}})$ preserves J and preserves the Pfaffian system of Cartan's claim (4). The distribution D defined by this system is, upon restriction to the null cone $\{J = 0\} \setminus \{0\}$, precisely the distribution D which we defined in the final section of our paper: $D(\mathbf{x}, \mathbf{y}, z) := \{(\mathbf{a}, \mathbf{b}, c) : (\mathbf{x}, \mathbf{y}, z)(\mathbf{a}, \mathbf{b}, c) = 0\}$. It follows that Cartan's construction, pushed down to the space of rays using the $\mathbf{R}^+$-action, yields precisely our $\widetilde{Q}$.

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