

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 55 (2009)
Heft: 1-2

Artikel: A contact geometric proof of the Whitney-Graustein theorem
Autor: Geiges, Hansjörg
DOI: <https://doi.org/10.5169/seals-110096>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 11.07.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

A CONTACT GEOMETRIC PROOF OF
THE WHITNEY–GRAUSTEIN THEOREM

by Hansjörg GEIGES *)

ABSTRACT. The Whitney–Graustein theorem states that regular closed curves in the 2-plane are classified, up to regular homotopy, by their rotation number. Here we give a simple proof based on contact geometry.

1. INTRODUCTION

A *regular closed curve* in the 2-plane is a continuously differentiable map $\bar{\gamma}: [0, 2\pi] \rightarrow \mathbf{R}^2$ with the following properties:

- (i) $\bar{\gamma}(0) = \bar{\gamma}(2\pi), \quad \bar{\gamma}'(0) = \bar{\gamma}'(2\pi),$
- (ii) $\bar{\gamma}'(s) \neq \mathbf{0}$ for all $s \in [0, 2\pi]$.

If we identify the circle S^1 with $\mathbf{R}/2\pi\mathbf{Z}$, we may think of $\bar{\gamma}$ as a continuously differentiable map $S^1 \rightarrow \mathbf{R}^2$.

The *rotation number* $\text{rot}(\bar{\gamma})$ of $\bar{\gamma}$ is the degree of the map

$$\begin{aligned} S^1 &\longrightarrow \mathbf{R}^2 \setminus \{\mathbf{0}\}, \\ s &\longmapsto \bar{\gamma}'(s). \end{aligned}$$

*) The author is partially supported by DFG grant GE 1245/1-2 within the framework of the Schwerpunktprogramm 1154 “Globale Differentialgeometrie”.

In other words, $\text{rot}(\bar{\gamma})$ is simply a signed count of the number of complete turns of the velocity vector $\bar{\gamma}'$ as we once traverse the closed curve $\bar{\gamma}$, see Figure 1.

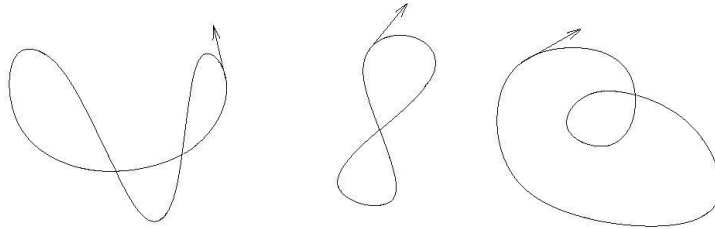


FIGURE 1

Regular closed curves $\bar{\gamma}$ with $\text{rot}(\bar{\gamma})$ equal to 1, 0, -2 , respectively

A *regular homotopy* between two such regular closed curves $\bar{\gamma}_0, \bar{\gamma}_1$ is a continuously differentiable homotopy via regular closed curves $\bar{\gamma}_t: S^1 \rightarrow \mathbf{R}^2$, $t \in [0, 1]$. The rotation number clearly stays invariant under regular homotopies. The following theorem is commonly known as the Whitney–Graustein theorem. It was first proved in a paper by H. Whitney [5], who writes: “This theorem, together with its proof, was suggested to me by W.C. Graustein.” For alternative presentations see [1, Chapter 0] or [3, p.47 *et seq.*].

THEOREM 1. *Regular homotopy classes of regular closed curves $\bar{\gamma}: S^1 \rightarrow \mathbf{R}^2$ are in one-to-one correspondence with the integers, the correspondence being given by $[\bar{\gamma}] \mapsto \text{rot}(\bar{\gamma})$.*

Whitney’s proof is elementary, but not without intricacies. Here we want to present a non-elementary proof — based on contact geometry — where the geometric ideas are actually quite simple.

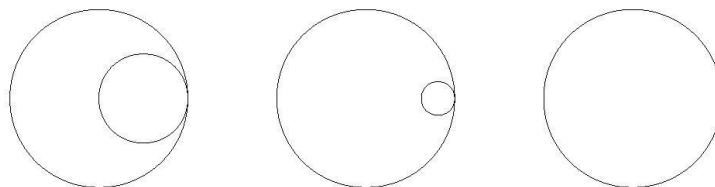


FIGURE 2

A homotopy through regular closed curves with non-invariant rot

REMARK. The modern terminology ‘regular homotopy’ describes what Whitney called a ‘deformation’ of regular closed curves. He seems to suggest, erroneously, that it is enough to require that $\gamma_t(s)$ be continuous in s and t and a regular closed curve for each fixed t , but in the course of his argument it becomes clear that he also wants $\gamma'_t(s)$ to depend continuously on t . Figure 2 shows a homotopy of regular closed curves (first traverse the big circle counter-clockwise, then the small circle) with $\text{rot}(\gamma_t) = 2$ for $t \in [0, 1)$, but $\text{rot}(\gamma_1) = 1$.

2. LEGENDRIAN CURVES

The *standard contact structure* ξ on \mathbf{R}^3 , see Figure 3 (produced by Stephan Schönenberger), is the 2-plane field $\xi = \ker(dz + x dy)$. For a brief introduction to contact geometry see [2]. No knowledge of contact geometry beyond the concepts that we shall introduce explicitly will be required for the argument that follows.

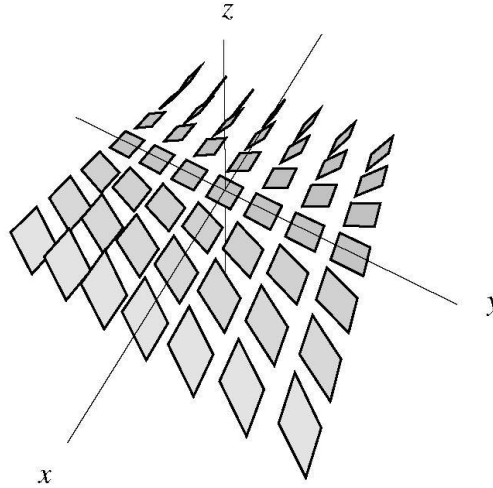


FIGURE 3

The contact structure $\xi = \ker(dz + x dy)$

A regular closed, continuously differentiable curve $\gamma: S^1 \rightarrow (\mathbf{R}^3, \xi)$ is called *Legendrian* if it is everywhere tangent to ξ , that is, $\gamma'(s) \in \xi_{\gamma(s)}$ for all $s \in S^1$. When we write γ in terms of coordinate functions as $\gamma(s) = (x(s), y(s), z(s))$, the condition for γ to be Legendrian becomes $z' + xy' \equiv 0$. The *front projection* of γ is the planar curve

$$\gamma_F(s) = (y(s), z(s)) ;$$

its *Lagrangian projection*, the curve

$$\gamma_L(s) = (x(s), y(s)) .$$

Figure 4 shows the front and Lagrangian projection of a Legendrian unknot in \mathbf{R}^3 .

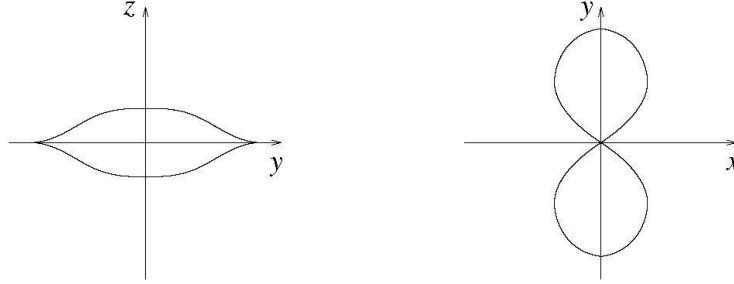


FIGURE 4
A Legendrian unknot

Notice that a Legendrian curve γ can be recovered from its front projection γ_F , since

$$x(s) = -\frac{z'(s)}{y'(s)} = -\frac{dz}{dy}$$

is simply the negative slope of the front projection. (Of course this only makes sense for $y'(s) \neq 0$. Generically, the zeros of the function $y'(s)$ are isolated, corresponding to isolated cusp points where γ_F still has a well-defined slope.) Since $x(s)$ is always finite, γ_F does not have any vertical tangencies, and we can sensibly speak of left and right cusps. These cusps are ‘semi-cubical’; a model is given by $(x(s), y(s), z(s)) = (s, s^2/2, -s^3/3)$.

Likewise, γ can be recovered from its Lagrangian projection γ_L (unique up to translation in the z -direction), for the missing coordinate z is given by

$$z(s_1) = z(s_0) - \int_{s_0}^{s_1} x(s)y'(s) ds .$$

Observe that the integral $\int xy' ds = \int x dy$, when integrating over a closed curve, measures the oriented area enclosed by that curve. Moreover, the Lagrangian projection γ_L of a regular Legendrian curve γ is always regular: if $y'(s) = 0$, the Legendrian condition forces $z'(s) = 0$, and then the regularity of γ gives $x'(s) \neq 0$.

The idea for the proof of Theorem 1 is now the following. Given a (regular closed) Legendrian curve γ in (\mathbf{R}^3, ξ) , one can assign to it an invariant

(under Legendrian regular homotopies, i.e. regular homotopies via Legendrian curves). This invariant is likewise called ‘rotation number’. In fact, the rotation number of γ will be seen to equal the rotation number of its Lagrangian projection γ_L . Alternatively, the rotation number of γ can be computed from its front projection γ_F , where it becomes a simple combinatorial quantity (a count of cusps). Now, given two regular closed curves $\bar{\gamma}_0, \bar{\gamma}_1$ in the plane with equal rotation number, we can consider their lifts to Legendrian curves γ_0, γ_1 (still with equal rotation number), and in the front projection we can now ‘see’, in a combinatorial way, a Legendrian regular homotopy between them. The Lagrangian projection of this Legendrian regular homotopy will give us the regular homotopy between $\bar{\gamma}_0$ and $\bar{\gamma}_1$.

3. THE ROTATION NUMBER

The plane field ξ is spanned by the globally defined vector fields $e_1 = \partial_x$ and $e_2 = \partial_y - x\partial_z$. In terms of the trivialisation of ξ defined by these vector fields, we may regard the map γ' (coming from a regular closed Legendrian curve γ) as a map

$$\begin{aligned} S^1 &\longrightarrow \mathbf{R}^2 \setminus \{\mathbf{0}\}, \\ s &\longmapsto \gamma'(s). \end{aligned}$$

The *rotation number* $\text{rot}(\gamma)$ of a Legendrian curve γ is the degree of that map. This means that $\text{rot}(\gamma)$ counts the number of rotations of the velocity vector γ' relative to the oriented basis e_1, e_2 of ξ as we go once around γ . The rotation number is clearly an invariant of Legendrian regular homotopies.

Under the projection $(x, y, z) \mapsto (x, y)$, each 2-plane $\xi_{\gamma(s)}$ maps isomorphically onto \mathbf{R}^2 , and the basis e_1, e_2 for $\xi_{\gamma(s)}$ is mapped to the standard basis ∂_x, ∂_y for \mathbf{R}^2 . So the following proposition is immediate from the definitions.

PROPOSITION 2. *The rotation number of a (regular closed) Legendrian curve in (\mathbf{R}^3, ξ) equals the rotation number of its Lagrangian projection.* \square

A little more work is required to read off $\text{rot}(\gamma)$ from the front projection γ_F . This, however, is well worth the effort, because it turns the rotation number into a simple combinatorial quantity.

PROPOSITION 3. *Let γ be a (regular closed) Legendrian curve in (\mathbf{R}^3, ξ) . Write λ_+ or λ_- , respectively, for the number of left cusps of the front projection γ_F oriented upwards or downwards; similarly we write ρ_{\pm} for the number of right cusps with one or the other orientation. Finally, we write c_{\pm} for the total number of cusps oriented upwards or downwards, respectively. Then the rotation number of γ is given by*

$$\text{rot}(\gamma) = \lambda_- - \rho_+ = \rho_- - \lambda_+ = \frac{1}{2}(c_- - c_+).$$

Proof. The rotation number $\text{rot}(\gamma)$ can be computed by counting (with sign) how often the velocity vector γ' crosses $e_1 = \partial_x$ as we travel once along γ .

Since $x(s)$ equals the negative slope of the front projection, points of γ where the (positive) tangent vector equals ∂_x are exactly the left cusps oriented downwards (see Figure 5) and the right cusps oriented upwards.

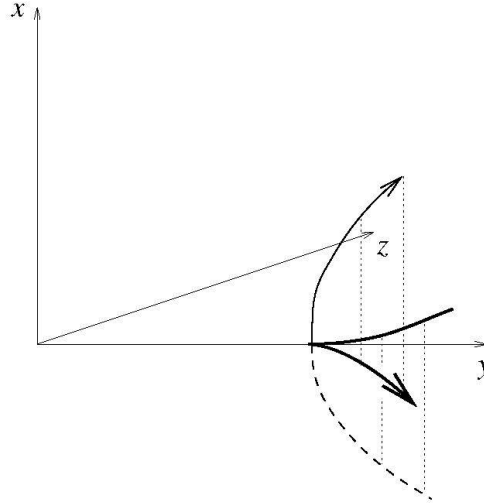


FIGURE 5

Contribution of a cusp to $\text{rot}(\gamma)$

At a left cusp oriented downwards, the tangent vector to γ , expressed in terms of e_1, e_2 , changes from having a negative component in the e_2 -direction to a positive one, i.e. such a cusp yields a positive contribution to $\text{rot}(\gamma)$. Analogously, one sees that a right cusp oriented upwards gives a negative contribution to the rotation number. This proves the formula $\text{rot}(\gamma) = \lambda_- - \rho_+$. The second expression for the rotation number is obtained by counting crossings through $-e_1$ instead; the third expression is found by averaging the first two. \square

4. PROOF OF THE WHITNEY–GRAUSTEIN THEOREM

First we give a classification of regular closed Legendrian curves up to Legendrian regular homotopy.

PROPOSITION 4. *Legendrian regular homotopy classes of regular closed Legendrian curves $\gamma: S^1 \rightarrow (\mathbf{R}^3, \xi)$ are in one-to-one correspondence with the integers, the correspondence being given by $[\gamma] \mapsto \text{rot}(\gamma)$.*

Proof. With the help of either of the two foregoing propositions one can construct a regular closed Legendrian curve γ with $\text{rot}(\gamma)$ equal to any prescribed integer. Thus, we need only show that two regular closed Legendrian curves $S^1 \rightarrow (\mathbf{R}^3, \xi)$ with the same rotation number are Legendrian regularly homotopic.

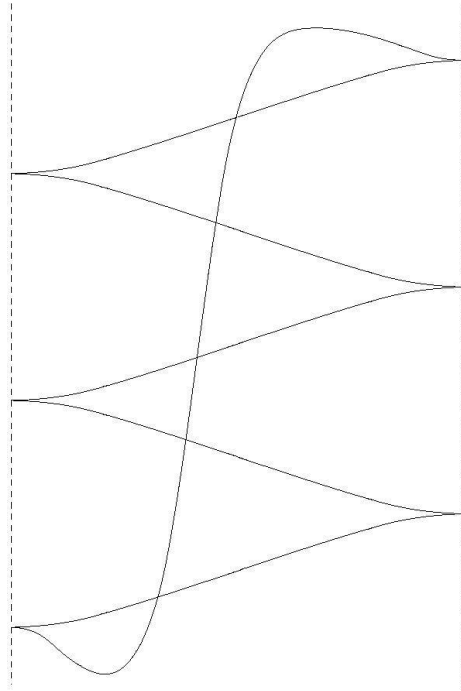


FIGURE 6

A front with cusps of one sign only

In the front projection of the Legendrian immersion γ , left and right cusps alternate. We label the up cusps with $+$ and the down cusps with $-$. The following observation will be crucial to our discussion.

CLAIM. *Up to Legendrian regular homotopy, γ is completely determined by this sequence of labels, starting at a right cusp, say, and going once around S^1 .*

This can be seen by homotoping γ_F so that all left cusps come to lie on the line $\{y = 0\}$ and all right cusps on the line $\{y = 1\}$, say. The cusps on either line can be shuffled by further homotopies; in particular, they may be arranged along these lines in the same order in which they are traversed along the closed Legendrian curve. This provides a standard model for any given sequence of labels, and thus proves the claim. Figure 6 shows this standard model for a front γ_F containing cusps of one sign only.

Continuing with the proof of the proposition, our aim now is to simplify the sequence of labels. Given a pair $+ -$ in this sequence, we can cancel it (unless it constitutes the complete sequence) as follows. Arrange the adjacent vertices (by sliding them along the lines $\{y = 0\}$ and $\{y = 1\}$, respectively, as described before) in such a way that we have the situation on the right of Figure 7, then replace it by the situation on the left. This so-called *first Legendrian Reidemeister move* is in fact a Legendrian isotopy for that local piece of our curve, i.e. a regular homotopy not creating self-intersections. There is an analogous move with the picture rotated by 180° , which can be used to cancel any pair $- +$.

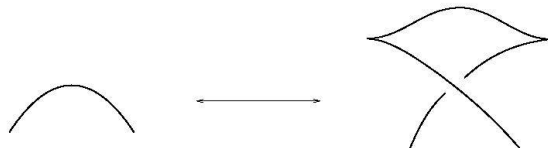


FIGURE 7

The first Legendrian Reidemeister move

Therefore, this sequence of labels can be reduced to a sequence containing only plus or only minus signs, or to one of the sequences $(+, -)$, $(-, +)$; see Figure 8 for an example. The formula $\text{rot}(\gamma) = (c_- - c_+)/2$ shows that there are the following possibilities: if $\text{rot}(\gamma)$ is positive (resp. negative), we must have a sequence of $2\text{rot}(\gamma)$ minus (resp. plus) signs; if $\text{rot}(\gamma) = 0$, we must have the sequence $(+, -)$ or $(-, +)$. The proof is completed by observing that these last two sequences correspond to Legendrian isotopic knots: use a first Reidemeister move as in Figure 7, followed by the inverse of the rotated move. \square

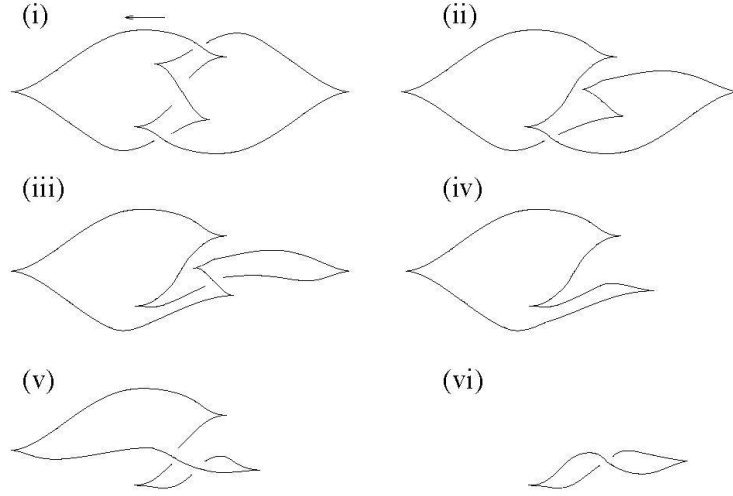


FIGURE 8

An example of a Legendrian regular homotopy

REMARK. Self-tangencies in the front projection γ_F correspond to self-intersections of the Legendrian curve γ , since the negative slope of γ_F gives the x -component of γ . Therefore, as we pass such a self-tangency in the moves of Figure 8, we effect a crossing change. With the orientation indicated in the figure, this example has $\text{rot}(\gamma) = -1$.

Proof of Theorem 1. Again we only have to show that two regular closed curves $\bar{\gamma}_0, \bar{\gamma}_1: S^1 \rightarrow \mathbf{R}^2$ (where we think of \mathbf{R}^2 as the (x, y) -plane) with $\text{rot}(\bar{\gamma}_0) = \text{rot}(\bar{\gamma}_1)$ are regularly homotopic.

After a regular homotopy we may assume that the $\bar{\gamma}_i$ satisfy the area condition $\oint_{\bar{\gamma}_i} x dy = 0$ and thus lift to regular *closed* Legendrian curves $\gamma_i: S^1 \rightarrow (\mathbf{R}^3, \xi)$ with, by Proposition 2, $\text{rot}(\gamma_i) = \text{rot}(\bar{\gamma}_i)$. By the preceding proposition, γ_0 and γ_1 are Legendrian regularly homotopic. The Lagrangian projection of this homotopy gives a regular homotopy between the curves $\bar{\gamma}_0$ and $\bar{\gamma}_1$, since — as pointed out in Section 2 — the Lagrangian projection of a regular Legendrian curve is regular. \square

REMARK. See [4] for an application of the ideas in the present paper to the classification of loops tangent to the standard Engel structure on \mathbf{R}^4 .

ACKNOWLEDGEMENTS. The idea for the proof presented here was inspired by a conversation with Yasha Eliashberg.

REFERENCES

- [1] ADACHI, M. *Embeddings and Immersions*. Translated from the 1984 Japanese original by K. Hudson. Translations of Mathematical Monographs 124. Amer. Math. Soc., Providence, 1993.
- [2] GEIGES, H. A brief history of contact geometry and topology. *Expo. Math.* 19 (2001), 25–53.
- [3] — *h*-principles and flexibility in geometry. Mem. Amer. Math. Soc. 164 (2003), no. 779.
- [4] — Horizontal loops in Engel space. *Math. Ann.* 342 (2008), 291–296.
- [5] WHITNEY, H. On regular closed curves in the plane. *Compos. Math.* 4 (1937), 276–284.

(Reçu le 12 octobre 2007)

Hansjörg Geiges

Mathematisches Institut
Universität zu Köln
Weyertal 86–90
50931 Köln
Germany
e-mail: geiges@math.uni-koeln.de