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LINEAR FUNCTIONAL EQUATIONS AND SHAPIRO'S CONJECTURE

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ABSTRACT. We investigate the functional equation

$$\sum_{i=1}^{n} a_i(y) f_i(x + b_i(y)) = h(y) \qquad (x, y \in \mathbf{R}),$$

where a_i, f_i, and h are complex valued functions defined on $\mathbf{R}$, and $b_1, \dots, b_n$ are real valued functions such that $b_i - b_j$ is not constant on any interval. We prove that under mild regularity conditions (e.g., if $a_1, \dots, a_n$ are nonvanishing functions of bounded variation, $b_1, \dots, b_n$ are d-convex and $f_1, \dots, f_n$ are measurable) the functions $f_1, \dots, f_n$ must be exponential polynomials. We also show that the continuity of the functions b_i and f_i implies the same conclusion, subject to Shapiro's conjecture on exponential polynomials with constant coefficients.

1. INTRODUCTION

The functional equation

$$\sum_{i=1}^{n} a_i(y) f_i(x + b_i(y)) = h(y) \tag{1}$$

has been studied extensively, and several papers have been devoted to the regularity properties of the solutions $f_1, \dots, f_n$. In [12] and [1] it is shown that if the functions a_i and b_i are smooth enough and if $f_1, \dots, f_n$ are locally integrable then $f_1, \dots, f_n$ are necessarily C^∞ functions. In this paper we show that under mild regularity conditions on the functions a_i and b_i, the functions f_i must be exponential polynomials, even if we only assume measurability instead of local integrability.

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We shall say that the function $\phi\colon [a,b] \to \mathbf{R}$ is *d-convex* if it can be written as the difference of two continuous convex functions. It is easy to see that $\phi\colon [a,b] \to \mathbf{R}$ is d-convex and Lipschitz if and only if ϕ is absolutely continuous and if the function ϕ' (defined on the set of points where ϕ is differentiable) is of bounded variation. Clearly, every C^2 function is d-convex.

A function $f\colon \mathbf{R} \to \mathbf{C}$ is said to be an exponential polynomial if $f(x) = \sum_{i=1}^{n} p_i(x)\, e^{\alpha_i x}$, where $p_1,\dots,p_n$ are polynomials with complex coefficients and $\alpha_1,\dots,\alpha_n$ are complex numbers.

THEOREM 1. *Let J be a nondegenerate interval, and suppose that the functions $a_i\colon J \to \mathbf{C}$ and $b_i\colon J \to \mathbf{R}$ $(i = 1,\dots,n)$ have the following properties.*

(i) *Each of the functions $a_1,\dots,a_n$ is nonvanishing on J and is of bounded variation;*

(ii) *each of the functions $b_1,\dots,b_n$ is d-convex on J; and*

(iii) *the function $b_i - b_j$ is not constant on any subinterval of J for every $1 \le i < j \le n$.*

Let $h\colon J \to \mathbf{C}$ be an arbitrary function, and let $f_1,\dots,f_n$ be complex valued measurable functions on $\mathbf{R}$ such that (1) *holds for almost every $(x,y) \in \mathbf{R}\times J$. Then each of the functions $f_1,\dots,f_n$ equals an exponential polynomial almost everywhere.*

The necessity of condition (iii) is shown by the fact that any function $f\colon \mathbf{R} \to \mathbf{C}$ satisfies

$$f(x) + f(x+y) - f(x+\max(y,0)) - f(x+\min(y,0)) = 0$$

for every $(x,y) \in \mathbf{R}^2$.

We can formulate many similar statements by imposing different conditions on the functions involved. Two of the most interesting variants are the following.

STATEMENT M. *Suppose that the functions $a_i\colon J \to \mathbf{C}$ and $b_i\colon J \to \mathbf{R}$ $(i = 1,\dots,n)$ are measurable, a_i is nonvanishing on J for every $i = 1,\dots,n$, and $b_i - b_j$ is not constant on any set of positive measure for every $1 \le i < j \le n$. Let $h\colon J \to \mathbf{C}$ be an arbitrary function, and let $f_1,\dots,f_n$ be complex valued measurable functions on $\mathbf{R}$ such that* (1) *holds for almost every $(x,y) \in \mathbf{R} \times J$. Then each of the functions $f_1,\dots,f_n$ equals an exponential polynomial almost everywhere.*

STATEMENT C. *Suppose that the functions* $a_i: J \to \mathbf{C}$ *and* $b_i: J \to \mathbf{R}$ $(i = 1, \dots, n)$ *are continuous,* a_i *is nonvanishing on* J *for every* $i = 1, \dots, n$, *and* $b_i - b_j$ *is not constant on any subinterval of* J *for every* $1 \le i < j \le n$. *Let* $h: J \to \mathbf{C}$ *be an arbitrary function, and let* $f_1, \dots, f_n$ *be complex valued continuous functions on* $\mathbf{R}$ *such that* (1) *holds for every* $(x, y) \in \mathbf{R} \times J$. *Then each of the functions* $f_1, \dots, f_n$ *is an exponential polynomial.*

We do not know if Statements M and C are true or not. We shall prove, however, that Statement C is a consequence of Shapiro's conjecture.

Let $\mathcal{R}$ denote the set of difference operators of the form

$$\Delta f = \sum_{i=1}^{n} a_i \cdot f(x + b_i),$$

where a_i and b_i are complex. If we define addition in the obvious way and multiplication by $(\Delta_1 \Delta_2) f = \Delta_1 (\Delta_2 f)$ then we obtain a commutative ring with identity. (In fact, what we obtain is the complex group ring over the additive group of $\mathbf{C}$.) The one-to-one correspondence between Δ and its characteristic function

$$\sum_{i=1}^{n} a_i e^{b_i z} \tag{2}$$

is an isomorphism between $\mathcal{R}$ and the ring $\mathcal{E}$ of all exponential polynomials with constant coefficients. The units of the ring $\mathcal{E}$ are the functions of the form $a \cdot e^{bz}$, where $a \neq 0$. The exponential polynomial (2) is called simple if the frequencies $b_1, \dots, b_n$ are pairwise commensurable; that is, if b_i / b_j is rational whenever $b_j \neq 0$. By a theorem of J. F. Ritt [9], every nonzero and non-unit exponential polynomial has a factorization of the form $f_1 \cdot \ldots \cdot f_s \cdot g_1 \cdot \ldots \cdot g_t$, where $f_1, \dots, f_s$ are simple, the frequencies of f_i and f_j are noncommensurable if $i \neq j$, and each g_k is irreducible. The factorization is unique up to unit multiples.

H. S. Shapiro conjectured in [11] that if two exponential polynomials have infinitely many common roots then they have a non-unit common divisor. As Shapiro remarked, the Lech-Mahler theorem implies the conjecture in the special case when one of the exponential polynomials is simple. (See [11, p. 18] and [8].) The conjecture in its general form is still open.

Recall that a topological space Y is Baire if every meager subset of Y has empty interior.

THEOREM 2. *Suppose that Shapiro's conjecture is true. Let Y be a topological space such that Y^n is Baire, and let the functions $a_i\colon Y \to \mathbf{C}$ and $b_i\colon Y \to \mathbf{R}$ $(i = 1, \dots, n)$ satisfy the following conditions: a_i is nonvanishing on Y, b_i is continuous for every $i = 1, \dots, n$, and $b_i - b_j$ is not constant on any nonempty open subset of Y for every $1 \le i < j \le n$. Let $h\colon Y \to \mathbf{C}$ be an arbitrary function, and let $f_1, \dots, f_n$ be complex valued continuous functions on $\mathbf{R}$ such that* (1) *holds for every $(x, y) \in \mathbf{R} \times Y$. Then each of the functions $f_1, \dots, f_n$ is an exponential polynomial.*

2. TRANSLATION INVARIANT CLOSED SUBSPACES OF $C(\mathbf{R})$

Let $C(\mathbf{R})$ denote the space of complex valued continuous functions on $\mathbf{R}$ endowed with the topology of uniform convergence on compact intervals. In the proof of Theorems 1 and 2 we shall use L. Schwartz's celebrated theorem stating that spectral synthesis holds in $C(\mathbf{R})$; that is, if L is any translation invariant closed subspace of $C(\mathbf{R})$ then the set of exponential polynomials contained in L form a dense subset of L. (See [10], [5] and [6].) Schwartz's theorem immediately implies that if L is a finite dimensional invariant subspace of $C(\mathbf{R})$ then L consists of exponential polynomials. We prove Theorem 1 – at least in the case when $h \equiv 0$ – by showing that the functions f_i must belong to finite dimensional invariant subspaces of $C(\mathbf{R})$.

LEMMA 3. *Let L be a translation invariant closed subspace of $C(\mathbf{R})$. Suppose that*

(i) *there exists a nonzero difference operator Δ such that $\Delta f = 0$ for every $f \in L$, and*

(ii) *every element of L is locally Lipschitz.*

Then L is finite dimensional.

Proof. Let $\Delta f(x) = \sum_{j=1}^{p} a_j f(x + b_j)$ $(f \in C(\mathbf{R}))$, where $a_1, \dots, a_p$ are nonzero and $b_1 < \dots < b_p$. If L is not finite dimensional then, by Schwartz's theorem, the spectrum $\mathrm{sp}(L) = \{\lambda \in \mathbf{C} : e^{\lambda x} \in L\}$ is infinite. If $\lambda \in \mathrm{sp}(L)$ then $\Delta e^{\lambda z} = 0$ by (i), and thus $E(\lambda) = 0$, where $E(z) = \sum_{j=1}^{p} a_j e^{b_j z}$. That is, $\mathrm{sp}(L)$ is a subset of the set of roots of $E(z)$, and hence the elements of $\mathrm{sp}(L)$ can be listed as $\lambda_n = \sigma_n + i t_n$ $(n = 1, 2, \dots)$, where $|\lambda_n| \to \infty$. Now

$$\lim_{\operatorname{Re} z \to \infty} \frac{E(z)}{e^{b_p z}} = a_p \qquad \text{and} \qquad \lim_{\operatorname{Re} z \to -\infty} \frac{E(z)}{e^{b_1 z}} = a_1 ,$$

and hence there is a positive number K such that $E(\sigma + it) \neq 0$ if $|\sigma| > K$. Therefore $|\sigma_n| \leq K$ for every n. Since $|\lambda_n| \to \infty$, it follows that $|t_n| \to \infty$.

We select a sequence $n_1, n_2 \ldots$ as follows. Let n_1 be chosen such that $|t_{n_1}| > 20\pi K$. If $n_1, \ldots, n_{k-1}$ have been selected then we choose n_k with the following properties: $|t_{n_k}| > 20^k \pi K$, and

$$\left| \exp\left(\frac{\pi \lambda_{n_j}}{t_{n_k}} \right) - 1 \right| < \frac{1}{10^k} \tag{3}$$

for every $j < k$. This defines the indices n_k for every k. Now we put $f(x) = \sum_{j=1}^{\infty} 10^{-j} e^{\lambda_{n_j} x}$ for every $x \in \mathbf{R}$. Since $\left| e^{\lambda_n x} \right| \leq e^{K \cdot |x|}$ for every n and for every $x \in \mathbf{R}$, it follows that the series is uniformly convergent on compact intervals, and thus f is an element of L. We shall prove that f is not locally Lipschitz at 0. By (ii), this will provide a contradiction, proving that sp(L) must be finite.

We have $f(\pi/t_{n_k}) - f(0) = \sum_{j=1}^{\infty} 10^{-j} A_k^j$, where

$$A_k^j = \exp\left(\frac{\pi \sigma_{n_j} + i\pi t_{n_j}}{t_{n_k}} \right) - 1 .$$

Now $\left| A_k^j \right| < 10^{-k}$ for every $j < k$ by (3),

$$|A_k^k| = \left| \exp\Big(\frac{\pi \sigma_{n_k}}{t_{n_k}} + i\pi \Big) - 1 \right| = \exp\Big(\frac{\pi \sigma_{n_k}}{t_{n_k}} \Big) + 1 > 1 ,$$

and

$$|A_k^j| \leq \exp\Big(\frac{\pi \sigma_{n_j}}{t_{n_k}} \Big) + 1 \leq \exp\Big(\frac{\pi K}{t_{n_k}} \Big) + 1 < 3$$

for every $j > k$. Therefore,

$$\begin{aligned} |f(\pi/t_{n_k}) - f(0)| &\geq \frac{1}{10^k} |A_k^k| - \sum_{j=1}^{k-1} \frac{1}{10^j} |A_k^j| - \sum_{j=k+1}^{\infty} \frac{1}{10^j} |A_k^j| \\ &\geq \frac{1}{10^k} - \sum_{j=1}^{k-1} \frac{1}{10^j} \cdot \frac{1}{10^k} - \sum_{j=k+1}^{\infty} \frac{1}{10^j} \cdot 3 \\ &\geq \frac{1}{2 \cdot 10^k} . \end{aligned}$$

Thus

$$\left| \frac{f(\pi/t_{n_k}) - f(0)}{(\pi/t_{n_k})} \right| \geq \frac{1}{2 \cdot 10^k} \cdot \frac{20^k \pi K}{\pi} = 2^{k-1} K$$

for every k, proving that f is not locally Lipschitz. $\square$

REMARK. Condition (i) cannot be omitted from Lemma 3: there are infinite dimensional translation invariant closed subspaces of $C(\mathbf{R})$ that only contain locally Lipschitz functions. One can show, for example, that if λ_n is a sequence of real numbers converging to infinity fast enough, then every element of the closed subspace L generated by the exponentials $e^{\lambda_n x}$ is real analytic, but L is not finite dimensional.

3. REDUCTION

Let G be an Abelian group, and let $\mathcal{R}_G$ denote the algebra of difference operators of the form $\Delta f = \sum_{i=1}^{n} a_i \cdot f(x+b_i)$ $(a_i \in \mathbf{C},\ b_i \in G)$. The translation operator T_b $(b \in G)$ is defined by $T_b f = f(x+b)$. Clearly, every difference operator is the linear combination of translation operators. We shall use determinants of the form

$$\begin{vmatrix} \Delta_{1,1} & \dots & \Delta_{1,n-1} & f_1 \\ \vdots & & \vdots & \vdots \\ \Delta_{n,1} & \dots & \Delta_{n,n-1} & f_n \end{vmatrix}, \tag{4}$$

where $\Delta_{i,j} \in \mathcal{R}_G$ $(i = 1,\dots,n;\quad j = 1,\dots,n-1)$, and $f_i \colon G \to \mathbf{C}$ $(i = 1,\dots,n)$. These determinants are defined as follows. In the formal expansion of (4) every term is of the form $\pm p_1 \cdots p_n$, where exactly one of the factors p_i is a function and the other factors are difference operators. Rearranging the factors such that the function comes last we obtain an expression of the form Δf, defining a map from G into $\mathbf{C}$. Then we define (4) as the sum of these functions.

Let Y be a nonempty set, and suppose that the functions $f_j \colon G \to \mathbf{C}$, $a_j \colon Y \to \mathbf{C}$, $b_j \colon Y \to G$ $(j = 1,\dots,n)$ and $h \colon Y \to \mathbf{C}$ satisfy

$$\sum_{j=1}^{n} a_j(y) \cdot f_j(x + b_j(y)) = h(y) \tag{5}$$

for every $(x,y) \in G \times Y$. We can write (5) as

$$\sum_{j=1}^{n} a_j(y) T_{b_j(y)} f_j = h(y)\,. \tag{6}$$

Let $y_1,\dots,y_n \in Y$ be arbitrary elements. Substituting $y_1,\dots,y_n \in Y$ into (6) we obtain $\sum_{j=1}^{n} a_j(y_i) T_{b_j(y_i)} f_j = h(y_i)$ $(i = 1,\dots,n)$.

Then we have

$$
(7)\qquad \begin{vmatrix} a_1(y_1)T_{b_1(y_1)} & \dots & a_{n-1}(y_1)T_{b_{n-1}(y_1)} & \sum_{j=1}^n a_j(y_1)T_{b_j(y_1)}f_j \\ \vdots & & \vdots & \vdots \\ a_1(y_n)T_{b_1(y_n)} & \dots & a_{n-1}(y_n)T_{b_{n-1}(y_n)} & \sum_{j=1}^n a_j(y_n)T_{b_j(y_n)}f_j \end{vmatrix}
$$
$$
= \begin{vmatrix} a_1(y_1)T_{b_1(y_1)} & \dots & a_{n-1}(y_1)T_{b_{n-1}(y_1)} & a_n(y_1)T_{b_n(y_1)}f_n \\ \vdots & & \vdots & \vdots \\ a_1(y_n)T_{b_1(y_n)} & \dots & a_{n-1}(y_n)T_{b_{n-1}(y_n)} & a_n(y_n)T_{b_n(y_n)}f_n \end{vmatrix};
$$

this can be justified in the same way as for determinants with numerical entries. The left hand side of (7), as a function of x, is constant, since each entry of its last column is constant. If we denote the value of the left hand side by $H(y) = H(y_1, \dots, y_n)$ and expand the right hand side of (7), then we obtain the following

LEMMA 4. *Suppose that the functions $f_j\colon G \to \mathbf{C}$, $a_j\colon Y \to \mathbf{C}$, $b_j\colon Y \to G$ $(j = 1, \dots, n)$ and $h\colon Y \to \mathbf{C}$ satisfy* (5) *for every $(x, y) \in G \times Y$. Put $N = n!$. Then there are functions $A_i\colon Y^n \to \mathbf{C}$ and $B_i\colon Y^n \to G$ $(i = 1, \dots, N)$ and $H\colon Y^n \to \mathbf{C}$ such that*

(i) *we have*

$$
(8)\qquad \sum_{i=1}^{N} A_i(y) f_n(x + B_i(y)) = H(y)
$$

for every $x \in G$ and $y \in Y^n$;

(ii) *for every $i = 1, \dots, N$ there are indices $j_1, \dots, j_n$ such that $A_i(y) = \pm a_{j_1}(y_1) \cdots a_{j_n}(y_n)$ for every $y = (y_1, \dots, y_n) \in Y^n$;*

(iii) *for every $i = 1, \dots, N$ there are indices $k_1, \dots, k_n$ such that $B_i(y) = b_{k_1}(y_1) + \dots + b_{k_n}(y_n)$ for every $y = (y_1, \dots, y_n) \in Y^n$;*

(iv) *if $b_{j_1} - b_{j_2}$ is not constant for every $1 \le j_1 < j_2 \le n$, then $B_{i_1} - B_{i_2}$ is not constant for every $1 \le i_1 < i_2 \le N$;*

(v) *if $h \equiv 0$ then $H \equiv 0$.*

REMARK. We shall need the following 'almost everywhere' version of Lemma 4 in the special case when $G = \mathbf{R}$ and Y is a subinterval of $\mathbf{R}$. Suppose that the measurable functions $f_j\colon \mathbf{R} \to \mathbf{C}$, $a_j\colon Y \to \mathbf{C}$, $b_j\colon Y \to \mathbf{R}$ $(j = 1, \dots, n)$ and $h\colon Y \to \mathbf{C}$ satisfy (5) for a.e. $(x, y) \in \mathbf{R} \times Y$ with respect to the Lebesgue measure λ_2. Then there are functions $A_i\colon Y^n \to \mathbf{C}$ and $B_i\colon Y^n \to \mathbf{R}$ $(i = 1, \dots, N)$ and $H\colon Y^n \to \mathbf{C}$ satisfying (ii)–(v) of Lemma 4 and such that (8) holds for a.e. $(x,y) \in \mathbf{R} \times Y^n$ with respect to λ_{n+1}. The proof of this statement is the same as that of Lemma 4.

4. Regularity of solutions

In this section we show that – under the conditions formulated in Theorem 1 – the measurable solutions of (1) are locally Lipschitz. We remark that by imposing more restrictive regularity conditions on the functions a_i and b_i (namely, $a_i, b_i \in C^2$) this result could be deduced from a general theorem of A. Járai [4]. Our result is based on the observation that if f is bounded measurable and g is of bounded variation then their convolution is Lipschitz. (See Lemma 7 below.)

LEMMA 5. *If g is a nonconstant d-convex function on J then there are a subinterval $J_1 \subset J$ and a positive number ε such that g is strictly monotonic on J_1; moreover, either $g'(x) \geq \varepsilon$ for a.e. $x \in J_1$ or $g'(x) \leq -\varepsilon$ for a.e. $x \in J_1$.*

Proof. Since g is absolutely continuous and nonconstant, the set $H = \{x \in J : g'(x) \neq 0\}$ is of positive measure. Also, g' is of bounded variation in every closed subinterval of the interior of J, and thus g' is continuous almost everywhere. Consequently, there is a point $x_0 \in H$ at which g' is continuous. Let $0 < \varepsilon < |g'(x_0)|/2$ be fixed, and choose a small neighbourhood J_1 of x_0 such that $|g'(x) - g'(x_0)| < \varepsilon$ whenever $x \in J_1$ and g' exists. It is clear that J_1 and ε satisfy the requirements. □

LEMMA 6. *Let $g\colon J \to \mathbf{R}$ be differentiable a.e. on the bounded interval J, and suppose that $g'(x) \neq 0$ for a.e. $x \in J$. Then* (i) *$g^{-1}(H)$ is null for every null set $H \subset \mathbf{R}$, and* (ii) *for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\lambda(H) < \delta$ implies $\lambda(g^{-1}(H)) < \varepsilon$.*

Proof. Let $\lambda(H) = 0$, and suppose that $A = g^{-1}(H)$ is of positive outer measure. Since $g'(x) \neq 0$ for a.e. $x \in A$, we can select a positive number ε and a set $B \subset A$ of positive outer measure such that either $g'(x) > \varepsilon$ or $g'(x) < -\varepsilon$ for every $x \in B$. We may assume that $g' > \varepsilon$ on B, since otherwise we replace g by $-g$. Then there is a positive integer n and there is a subset $C \subset B$ of positive outer measure such that $(g(y) - g(x))/(y - x) > \varepsilon$ for every $x \in C$ and for every $y \in J$ with $0 < |y - x| < 1/n$. Let L be a subinterval of J such that $|L| < 1/n$ and $\lambda(C \cap L) > 0$. Put $D = C \cap L$; then $\lambda(D) > 0$ and $|g(y) - g(x)| \geq \varepsilon|y - x|$ for every $x, y \in D$. In particular, g is one-to-one on D. Let $g(D) = E$ and $f = (g \mid D)^{-1}$. Then $E \subset H$ and f maps E onto D. Also, f is Lipschitz on E, since $|f(u) - f(v)| \leq |u - v|/\varepsilon$ holds for every $u, v \in E$. Since $\lambda(E) \leq \lambda(H) = 0$, this implies $\lambda(D) = 0$, a contradiction. This proves (i).

Suppose that (ii) is false. Then there is an $\varepsilon > 0$ and there are sets H_n such that $\lambda(H_n) < 1/n^2$ and $\lambda\left(g^{-1}(H_n)\right) \geq \varepsilon$ for every $n = 1, 2, \ldots$. We may assume that the sets H_n are open. Since g is measurable (in fact, g is continuous a.e.), it follows that the sets $g^{-1}(H_n)$ are measurable. Let $H = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} H_n$. Then $\lambda(H) = 0$, and

$$\lambda\left(g^{-1}(H)\right) = \lambda\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} g^{-1}(H_n)\right) \geq \liminf_{n \to \infty} \lambda\left(g^{-1}(H_n)\right) \geq \varepsilon,$$

which contradicts (i). $\square$

LEMMA 7. *Let U be of bounded variation on the interval $[a, b]$. Let I be a compact interval, and let f be measurable and bounded on the interval $I + [a, b]$. Then the function*

$$F(x) = \int_a^b f(x + y) U(y)\, dy \qquad (x \in I)$$

is Lipschitz on I.

Proof. Let $I + [a, b] = [c, d]$, and put $\Phi(x) = \int_c^x f(t)\, dt$ $(x \in [c, d])$. Then Φ is a Lipschitz function such that $\Phi' = f$ a.e. on $I + [a, b]$. Denoting $\Phi(y + x)$ by $T_x\Phi(y)$ we obtain

$$\begin{aligned} (9) \qquad F(x) &= \int_a^b U \cdot (T_x\Phi)'\, dy = \int_a^b U\, d(T_x\Phi) = [U \cdot T_x\Phi]_a^b - \int_a^b T_x\Phi\, dU \\ &= U(b) \cdot \Phi(x + b) - U(a) \cdot \Phi(x + a) - \int_a^b T_x\Phi\, dU. \end{aligned}$$

If $|\Phi(x_1) - \Phi(x_2)| \leq K \cdot |x_1 - x_2|$ for every x_1, x_2 then we have

$$\begin{aligned} \left|\int_a^b T_{x_1}\Phi\, dU - \int_a^b T_{x_2}\Phi\, dU\right| &= \left|\int_a^b \left(T_{x_1}\Phi - T_{x_2}\Phi\right) dU\right| \\ &\leq K \cdot |x_1 - x_2| \cdot V(U; [a, b]), \end{aligned}$$

and thus the function $x \mapsto \int_a^b T_x\Phi\, dU$ is Lipschitz. Then, by (9), so is F. $\square$

LEMMA 8. *Suppose that*

$$F(x) = \int_a^b c(y) f(x + g(y))\, dy \qquad (x \in I),$$

where

- $c\colon [a,b] \to \mathbf{C}$ *is of bounded variation,*
- $g\colon [a,b] \to \mathbf{R}$ *is d-convex and Lipschitz,*
- *there is a positive number* ε *such that* $|g'(x)| \geq \varepsilon$ *at every point* $x \in [a,b]$ *where* $g'(x)$ *exists,*
- I *is a compact interval, and*
- f *is measurable and bounded on the interval* $I + g([a,b])$.

Then the function F *is Lipschitz on* I.

Proof. Since g' is of bounded variation, the oscillation of g' is less than 2ε everywhere, except at the points of a finite set. Then, by $|g'| \geq \varepsilon$ it follows that there is a subdivision $a = a_0 < a_1 < \ldots < a_n = b$ of $[a,b]$ such that, for every $i = 1, \ldots, n$, g is strictly monotonic on $[a_{i-1}, a_i]$, and either $g'(x) \geq \varepsilon$ for a.e. $x \in [a_{i-1}, a_i]$ or $g'(x) \leq -\varepsilon$ for a.e. $x \in [a_{i-1}, a_i]$. Let $F_i(x) = \int_{a_{i-1}}^{a_i} c(y) f(x+g(y))\, dy$ $(x \in I;\ i = 1, \ldots, n)$. Since $F = F_1 + \ldots + F_n$, it is enough to show that each F_i is Lipschitz on I.

Let i be fixed. We may assume that g is strictly increasing on $[a_{i-1}, a_i]$; the case when g is decreasing can be treated similarly. Let $A = g(a_{i-1})$, $B = g(a_i)$, and let G denote the inverse of $g \mid [a_{i-1}, a_i]$. Then G is absolutely continuous (in fact, Lipschitz) and strictly increasing on $[A,B]$. Since $G' = 1/(g' \circ G)$, $g' \geq \varepsilon$ and g' is of bounded variation on $[A,B]$, it follows that G' is also of bounded variation on $[A,B]$. Let U be an extension of $(c \circ G) \cdot G'$ to $[A,B]$ having finite variation. Then we have $F_i(x) = \int_A^B f(x+u) U(u)\, du$ for every $x \in I$, and thus, by Lemma 7, F_i is Lipschitz on I. □

For every closed interval J and positive integer n we shall denote by Φ_J^n the family of all functions of the form

$$A(y) = a_1(y_1) \cdots a_n(y_n) \qquad (y = (y_1, \ldots, y_n) \in J^n),$$

where $a_1, \ldots, a_n$ are complex valued nonvanishing functions of bounded variation defined on J. The set of the functions $b_1(y_1) + \ldots + b_n(y_n)$, where $b_i\colon J \to \mathbf{R}$ is a d-convex function on J for every $i = 1, \ldots, n$ will be denoted by Ψ_J^n.

By a subinterval of J^n we shall mean a set of the form $J_1 \times \ldots \times J_n$, where $J_1, \ldots, J_n$ are nondegenerate subintervals of J.

LEMMA 9. *Let $A_i \in \Phi_J^n$ and $B_i \in \Psi_J^n$ for every $i = 1, \ldots, N$, and suppose that $B_i - B_j$ is not constant on any subinterval of J^n for every $1 \leq i < j \leq N$. Let $f_1, \ldots, f_N$ be complex valued measurable functions on $\mathbf{R}$ such that*

$$\sum_{i=1}^{N} A_i(y) f_i(x + B_i(y)) = 0 \tag{10}$$

for almost every $(x, y) \in \mathbf{R} \times J^n$. Then each of the functions $f_1, \ldots, f_N$ equals a locally Lipschitz function almost everywhere.

Proof. By symmetry, it is enough to show that f_1 equals a locally Lipschitz function almost everywhere.

Let U denote the set of points $(x, y) \in \mathbf{R} \times J^n$ for which (10) holds. Then $(x - B_1(y), y) \in U$ for a.e. $(x, y) \in \mathbf{R} \times J^n$, and thus

$$\sum_{i=1}^{N} A_i(y) f_i(x + B_i(y) - B_1(y)) = 0$$

holds for a.e. $(x, y) \in \mathbf{R} \times J^n$. Therefore we may replace B_i by $B_i - B_1$ for every i. After these replacements we find that $B_1 \equiv 0$.

Let $A_i(y) = \prod_{k=1}^{n} a_{i,k}(y_k)$ and $B_i(y) = \sum_{k=1}^{n} b_{i,k}(y_k)$, where $a_{i,k} \colon J \to \mathbf{C}$ is a nonvanishing function of bounded variation, and $b_{i,k} \colon J \to \mathbf{R}$ is a d-convex function for every $i = 1, \ldots, N$ and $k = 1, \ldots, n$. Since the functions $a_{i,k}$ are continuous everywhere on J apart from a countable set, they have a common point of continuity x_0. As $a_{i,k}(x_0) \neq 0$ for every i and k, there is an $\eta > 0$ and there is a neighbourhood J_0 of x_0 such that $|a_{i,k}(x)| > \eta$ for every $i = 1, \ldots, N$, $k = 1, \ldots, n$ and $x \in J_0$. Replacing J by J_0 we may clearly assume that $|a_{i,k}(x)| > \eta$ holds everywhere on J for every i and k. Then $a_{i,k}/a_{1,k}$ is of bounded variation for every i and k, and thus $A_i/A_1 \in \Phi_J^n$ for every $i = 1, \ldots, N$. We replace A_i by A_i/A_1 for every i; then we have $A_1 \equiv 1$ and

$$f_1(x) = -\sum_{i=2}^{N} A_i(y) f_i(x + B_i(y)) \tag{11}$$

for a.e. $(x, y) \in \mathbf{R} \times J^n$.

Let $1 < i \leq N$ and the subinterval $J' \subset J$ be fixed. We claim that there is a $k \in \{1, \ldots, n\}$ and there is a subinterval $J'' \subset J'$ such that $b_{i,k}$ is not constant in every subinterval of J''. Indeed, otherwise we could find, successively, the intervals $J' \supset J_1 \supset J_2 \supset \ldots \supset J_n$ such that $b_{i,k}$ is constant in J_k for every $k = 1, \ldots, n$. Then $B_i = B_i - B_1$ would be constant in $(J_n)^n$, contrary to the assumption. Applying this observation for every $1 < i \le N$ successively, we find a subinterval $\bar{J} \subset J$ with the following property: for every $1 < i \le N$ there is a $k(i) \in \{1, \dots, n\}$ such that $b_{i,k(i)}$ is not constant in every subinterval of $\bar{J}$. Clearly, we may assume that $J = \bar{J}$. By taking another subinterval of J, we can suppose that each $b_{i,k}$ is Lipschitz in J.

Applying Lemma 5, $N-1$ times in succession, we find a positive ε and a subinterval $J_1 \subset J$ such that, for every $1 < i \le N$, $b_{i,k(i)}$ is strictly monotonic on J_1, and $\left|b'_{i,k(i)}\right| \ge \varepsilon$ almost everywhere on J_1. Again, we may assume that $J_1 = J$. Then, by Lemma 6, we can find a positive number δ such that $\lambda\left(b_{i,k(i)}^{-1}(H)\right) < |J|/N$ whenever $\lambda(H) < \delta$ and $i = 2, \dots, N$.

Let $i \in \{2, \dots, N\}$ be arbitrary. We show that $\lambda_n(B_i^{-1}(H)) < |J|^n/N$ for every $H \subset \mathbf{R}$, $\lambda(H) < \delta$. We may suppose that H is open, and then so is $B_i^{-1}(H)$. If $y_j \in J$ is fixed for every $j \in \{1, \dots, n\} \setminus \{k(i)\}$ then

$$(y_1, \dots, y_n) \in B_i^{-1}(H) \iff b_{i,k(i)}(y_{k(i)}) \in H - \sum_{j \ne k(i)} b_{i,j}(y_j),$$

and thus

$$\lambda(\{y_{k(i)} : (y_1, \dots, y_n) \in B_i^{-1}(H)\}) = \lambda\left(b_{i,k(i)}^{-1}\left(H - \sum_{j \ne k(i)} b_{i,j}(y_j)\right)\right) < |J|/N,$$

since $\lambda\left(H - \sum_{j \ne k(i)} b_{i,j}(y_j)\right) = \lambda(H) < \delta$. Therefore, by Fubini's theorem, we obtain

$$\lambda_n\left(B_i^{-1}(H)\right) < |J|^{n-1} \cdot |J|/N = |J|^n/N,$$

as we stated.

We prove that f_1 is locally essentially bounded. Let I be an arbitrary compact interval. Fubini's theorem implies that there is a set $X \subset \mathbf{R}$ of full measure such that for every $x \in X$, (11) holds for a.e. $y \in J^n$. If K is large enough then the measure of each of the sets $H_K^i = \{x \in I + B_i(J^n) : |f_i(x)| > K\}$ $(i = 2, \dots, N)$ is less than δ. Therefore, by the choice of δ, the set

$$E_x = \bigcup_{i=2}^{N} B_i^{-1}\left(H_K^i - x\right)$$

is of measure less than $|J|^n$ for every x. Then the set $J^n \setminus E_x$ is of positive measure for every $x \in \mathbf{R}$, and hence we can choose a point $y_x \in J^n \setminus E_x$ for every $x \in X$ such that (11) holds with $y = y_x$. Since $x + B_i(y_x) \notin H_K^i$ for every $i = 2, \dots, N$, we have

$$|f_1(x)| \le \sum_{i=2}^{N} \sup_{J^n} |A_i| \cdot K$$

for every $x \in I \cap X$. Since the interval I was arbitrary, it follows that f_1 is locally essentially bounded. Clearly, the same is true for every f_i.

Now we show that f_1 equals a locally Lipschitz function almost everywhere. By (11) we have

$$|J|^n \cdot f_1(x) = -\sum_{i=2}^{N} \int_{J^n} A_i(y) f_i(x + B_i(y))\, d\lambda_n(y)$$

for a.e. x. Clearly, it is enough to show that

$$F_i(x) = \int_{J^n} A_i(y) f_i(x + B_i(y))\, d\lambda_n(y) \qquad (x \in \mathbf{R})$$

defines a locally Lipschitz function for every $i = 2, \ldots, N$. Let i be fixed. Putting

$$u(z) = \prod_{j \neq k(i)} a_{i,j}(y_j) \qquad (z = (y_1, \ldots, y_{k(i)-1}, y_{k(i)+1}, \ldots, y_n))$$

we have

$$(12) \quad F_i(x) = \int_{J^{n-1}} u(z) \cdot \left[\int_J a_{i,k(i)}(t) \cdot f_i\left(x + d(z) + b_{i,k(i)}(t)\right) dt \right] d\lambda_{n-1}(z),$$

where $d(z) = \sum_{j \neq k(i)} b_{i,j}(y_j)$. By Lemma 8, the function

$$L(x) = \int_J a_{i,k(i)}(t) \cdot f_i(x + b_{i,k(i)}(t))\, dt$$

is locally Lipschitz on $\mathbf{R}$. Since

$$F_i(x) = \int_{J^{n-1}} u(z) \cdot L(x + d(z))\, d\lambda_{n-1}(z)$$

by (12), it follows that F_i is also locally Lipschitz. Indeed, let I be a compact interval. Since d is continuous on J^{n-1}, it follows that $I' = I + d(J^{n-1})$ is also a compact interval. Let K be the Lipschitz constant of L on I'. If $x_1, x_2 \in I$ and $z \in J^{n-1}$ then $x_1 + d(z)$, $x_2 + d(z) \in I'$ and thus

$$\begin{aligned} |F_i(x_2) - F_i(x_1)| &\leq \int_{J^{n-1}} |u(z)| \cdot |L(x_2 + d(z)) - L(x_1 + d(z))|\, d\lambda_{n-1}(z) \\ &\leq K \cdot |x_2 - x_1| \cdot \int_{J^{n-1}} |u(z)|\, d\lambda_{n-1}, \end{aligned}$$

proving that F_i is locally Lipschitz. $\square$

5. Proof of Theorem 1

First we shall assume that the function h is identically zero. By symmetry, it is enough to show that f_n equals an exponential polynomial almost everywhere.

Suppose that the functions a_i, b_i, f_i, and $h \equiv 0$ are as in Theorem 1. Applying the a.e.-version of Lemma 4, we find the functions $A_i\colon J^n \to \mathbf{C}$, $B_i\colon J^n \to \mathbf{R}$ $(i = 1, \dots, N)$ satisfying (ii)–(v) of Lemma 4 with $G = \mathbf{R}$ and $Y = J$ and such that (8) holds for a.e. $(x, y) \in \mathbf{R} \times J^n$.

By (iv) of Lemma 4, $B_i - B_j$ is not constant on any subinterval of J^n for every $i \neq j$. Therefore, by Lemma 9, f_n equals a locally Lipschitz function $\widetilde{f}_n$ almost everywhere.

By Fubini's theorem, there is a subset Y of J^n of full measure such that for every $y \in Y$, (8) holds for a.e. $x \in \mathbf{R}$. Since $f_n = \widetilde{f}_n$ a.e., it follows that, for every $y \in Y$, we have

$$\sum_{i=1}^{N} A_i(y)\widetilde{f}_n(x + B_i(y)) = 0 \tag{13}$$

for a.e. x. Then, by the continuity of the functions $\widetilde{f}_n$ and B_i we find that (13) holds for every $x \in \mathbf{R}$ and $y \in Y$.

Let L denote the set of continuous functions $f \in C(\mathbf{R})$ satisfying

$$\sum_{i=1}^{N} A_i(y)f(x + B_i(y)) = 0 \tag{14}$$

for every $(x, y) \in \mathbf{R} \times Y$. Then L is a translation invariant closed subspace of $C(\mathbf{R})$ and, by the argument above, $\widetilde{f}_n \in L$. If $f \in L$ then (14) holds for a.e. $(x, y) \in \mathbf{R} \times J^n$ and thus, by Lemma 9, f is locally Lipschitz. That is, each element of L is locally Lipschitz. We claim that there exists a nonzero difference operator Δ such that $\Delta f = 0$ for every $f \in L$. In fact, if $f \in L$ then we have $\Delta(y)f = 0$ for every $y \in Y$, where $\Delta(y) = \sum_{i=1}^{N} A_i(y)T_{B_i(y)}$. We have to show that $\Delta(y)$ is nonzero for at least one $y \in Y$. But this is clear, because $A_i(y) \neq 0$ for every $y \in J^n$, and $B_1(y), \dots, B_n(y)$ are distinct on a dense open subset of J^n.

Therefore we may apply Lemma 3. We find that L is finite dimensional, and thus each element of L is an exponential polynomial. Since $\widetilde{f}_n \in L$ and f_n equals $\widetilde{f}_n$ almost everywhere, this completes the proof, assuming $h \equiv 0$.

The general case can be reduced to the previous one as follows. It is enough to show that f_n equals an exponential polynomial almost everywhere.

Let Δ_b denote the difference operator defined by $\Delta_b f(x) = f(x+b) - f(x)$. Suppose that the functions a_i, b_i, f_i, and h are as in Theorem 1. Then we have

$$\sum_{i=1}^{n} a_i(y)\, \Delta_b f_i(x + b_i(y)) = 0$$

for almost every $(x, y) \in \mathbf{R} \times J$ and for every $b \in \mathbf{R}$. As we proved already, this implies that $\Delta_b f_n$ equals an exponential polynomial almost everywhere for every $b \in \mathbf{R}$. Then, in particular, $\Delta_b f_n$ equals a continuous function almost everywhere for each $b \in \mathbf{R}$. By a theorem of T. Keleti [7, Theorem 2.9] it follows that f_n equals a continuous function $\bar{f}_n$ almost everywhere. Since $\Delta_b \bar{f}_n$ equals an exponential polynomial almost everywhere and $\bar{f}_n$ is continuous, we find that $\Delta_b \bar{f}_n$ equals an exponential polynomial everywhere for every $b \in \mathbf{R}$. Therefore, by a theorem of F. W. Carroll [2], $\bar{f}_n$ is exponential polynomial, which completes the proof. □

6. Proof of Theorem 2

For every $E \in \mathcal{E}$ we shall denote by $\Lambda(E)$ the set of roots of E.

LEMMA 10. *Shapiro's conjecture implies that if* $\{E_j : j \in J\}$ *is a system of exponential polynomials with constant coefficients such that* $\bigcap_{j \in I} \Lambda(E_j)$ *is infinite, then either*

(i) *there is a non-unit exponential polynomial that divides each* E_j, *or*

(ii) *there is a nonzero complex number* γ *such that each* E_j *has a divisor of the form* $e^{r\gamma z} - c$, *where* $c \neq 0$ *and* $r \neq 0$ *is rational.*

Proof. By Ritt's theorem [9] we have $E_j = F_j \cdot G_j$ $(j \in J)$, where each F_j is the product of finitely many simple exponential polynomials, and each G_j is the product of finitely many irreducible factors. Let $\Lambda = \bigcap_{j \in J} \Lambda(E_j)$. Then $\Lambda \subset \Lambda(E_j) = \Lambda(F_j) \cup \Lambda(G_j)$ for every $j \in J$. Suppose that there exists a $j_0 \in J$ such that $\Lambda \cap \Lambda(G_{j_0})$ is infinite. Then there is an irreducible factor H of G_{i_0} such that $\Lambda \cap \Lambda(H)$ is infinite. Then $\Lambda(H) \cap \Lambda(E_j)$ is infinite for every $i \in J$, as it contains $\Lambda \cap \Lambda(H)$. If Shapiro's conjecture is true then H and E_j have a common non-unit factor. Since H is irreducible, this factor must be (a unit multiple of) H. Thus, in this case, H divides each E_j; that is, (i) holds.

Next suppose that $\Lambda \cap \Lambda(G_j)$ is finite for every $j \in J$. Then $\Lambda \cap \Lambda(F_j)$ must be infinite for every j. It is easy to see that if $F \in \mathcal{E}$ is simple then F is the product of a unit and of finitely many factors of the form $e^{az} - c$, where $a \neq 0$ and $c \neq 0$. Therefore, $\Lambda(F)$ is the union of finitely many arithmetical progressions (AP's).

Let $j_0 \in J$ be arbitrary. Since $\Lambda \cap \Lambda(F_{j_0})$ is infinite and $\Lambda(F_{j_0})$ is the union of finitely many AP's, there exists an arithmetical progression $A = \{b + nd : n \in \mathbf{Z}\}$ such that $\Lambda \cap A$ is infinite. Let $\gamma = d/(2\pi i)$. We show that every E_i has a divisor of the form $e^{r\gamma z} - c$, where $c \neq 0$ and $r \neq 0$ is rational. That is, in this case (ii) holds.

Let $j \in J$ be arbitrary. Since $\Lambda(G_j) \cap (\Lambda \cap A)$ is finite, there is a factor $e^{az} - c$ of F_j such that $\Lambda(e^{az} - c) \cap (\Lambda \cap A)$ is infinite. Now $\Lambda(e^{az} - c)$ is an AP with difference $(2\pi i)/a$, and thus $(2\pi i)/a$ and d must be commensurable; that is, $(2\pi i)/ad$ is rational. Thus $a/\gamma = r$ is rational, which completes the proof. $\square$

REMARK. As the following simple example shows, we cannot omit case (ii) from the statement of Lemma 10. Let G_n $(n = 1, 2, \ldots)$ be a sequence of non-associate irreducible exponential sums such that $\{1, \ldots, n!\} \subset \Lambda(G_n)$ for every n. Let $E_n = \left(e^{\frac{2\pi i}{n} z} - 1\right) \cdot G_n$ $(n = 1, 2, \ldots)$. It is easy to check that $\{n! : n = 1, 2, \ldots\} \subset \Lambda(E_n)$ for every n, but the E_n's do not have a common non-unit divisor.

Now we turn to the proof of Theorem 2. First we consider the case when the function h is identically zero. Suppose (1). Clearly, it is enough to show that f_n is an exponential polynomial. By Lemma 4, there are functions $A_i \colon Y^n \to \mathbf{C}$ and $B_i \colon Y^n \to \mathbf{R}$ $(i = 1, \ldots, N = n!)$ such that (8) holds for every $y = (y_1, \ldots, y_n) \in Y^n$, A_i is nonvanishing and B_i is continuous on Y^n for every i. Also, it follows from (iv) of Lemma 4 that $B_i - B_j$ is not constant on any nonempty open subset of Y^n for every $1 \leq i < j \leq N$. Consequently, there is a nonempty open set $U \subset Y^n$ such that $B_1(y), \ldots, B_N(y)$ are distinct and of the same order for every $y \in U$. We may assume that $B_1(y) < \ldots < B_N(y)$ $(y \in U)$.

Let L denote the set of functions $f \in C(\mathbf{R})$ satisfying

$$\sum_{i=1}^{N} A_i(y) f(x + B_i(y)) = 0 \tag{15}$$

for every $(x, y) \in \mathbf{R} \times Y^n$. Then L is a translation invariant closed subspace of $C(\mathbf{R})$, and $f_n \in L$. By Schwartz's theorem, it is enough to show that L is finite dimensional.

First we shall prove that the spectrum $\mathrm{sp}(L) = \{\lambda \in \mathbf{C} : e^{\lambda x} \in L\}$ is finite. Suppose $\lambda \in \mathrm{sp}(L)$. Then $\sum_{i=1}^N A_i(y)\, e^{\lambda B_i(y)} = 0$ for every $y \in Y^n$; that is, λ is a root of the exponential sum $E_y(z) = \sum_{i=1}^N A_i(y)\, e^{B_i(y)z}$ for every $y \in Y^n$. We prove, assuming Shapiro's conjecture, that the exponential sums E_y have only a finite number of common roots. Suppose this is not true. Then, by Lemma 10, one of the following two statements must be true:

(i) there is a non-unit exponential polynomial that divides each E_y, or

(ii) there is a nonzero complex number γ such that each E_y has a divisor of the form $e^{r\gamma z} - c$, where $c \neq 0$ and $r \neq 0$ is rational.

We show that each of these statements contradicts the condition that $B_i - B_j$ is not constant on nonempty open sets.

Suppose (i), and let $\sum_{i=1}^k \gamma_i e^{\delta_i z}$ be a non-unit exponential polynomial that divides each E_y. We may assume that $k \geq 2$, $\gamma_1, \dots, \gamma_k$ are nonzero, $\delta_1, \dots, \delta_k$ are distinct, and that $\delta_1 = 0$. Then we have

$$E_y(z) = \sum_{i=1}^{k} \gamma_i e^{\delta_i z} \cdot \sum_{j=1}^{m(y)} a_j(y)\, e^{\beta_j(y) z} \tag{16}$$

for every $y \in Y^n$, where $a_1(y), \dots, a_{m(y)}(y)$ are nonzero and $\beta_1(y), \dots, \beta_{m(y)}(y)$ are distinct for every y. By a theorem of Ritt, there is a complex number δ such that each of the numbers $\delta_i - \delta$ $(i = 1, \dots, k)$ and $\beta_j(y) + \delta$ $(j = 1, \dots, m(y))$ is a linear combination of $B_1(y), \dots, B_N(y)$ with rational coefficients. (See [9, p. 585] and [3, Lemma 2].) Since $B_i(y)$ is real for every i and y, it follows that $\delta_i - \delta$ and $\beta_j(y) + \delta$ are also real for every i, j and y. Now $\delta_1 = 0$ implies that δ is real, and thus δ_i and $\beta_j(y)$ are real for every $i = 1, \dots, k$, $j = 1, \dots, m(y)$ and $y \in Y^n$. We may assume that $0 = \delta_1 < \dots < \delta_k$.

Let $K(m) = \{y \in U : m(y) = m\}$ $(m = 1, 2, \dots)$. Then $U = \bigcup_{m=1}^{\infty} K(m)$. Since Y^n is a Baire space, it follows that $K(m)$ is not nowhere dense for at least one m. Fix such an m, and partition $K(m)$ into $m!$ subsets according to the ordering of the numbers $\beta_1(y), \dots, \beta_m(y)$. Then at least one of these subsets is not nowhere dense. In other words, there exists a non-nowhere dense subset K of $K(m)$ such that the ordering of the numbers $\beta_1(y), \dots, \beta_m(y)$ is the same for every $y \in K$. We may assume that $\beta_1(y) < \dots < \beta_m(y)$ $(y \in K)$. By (16) and $\delta_1 = 0$ we have $B_1(y) = \beta_1(y)$ for every $y \in K$.

Let J denote the set of those indices $j \in \{1, \dots, m\}$ for which $\beta_j - \beta_1$ is constant on a non-nowhere dense subset of K. Obviously, $1 \in J$. Let j_0 be the largest element of J, and let K_0 be a non-nowhere dense subset of K such that $\beta_{j_0} - \beta_1$ is constant on K_0. Put $K_i = \{y \in K_0 : \delta_k + \beta_{j_0}(y) = B_i(y)\}$ $(i = 2, \dots, N)$. If $y \in K_i$ then

$$B_i(y) - B_1(y) = \delta_k + \beta_{j_0}(y) - \beta_1(y),$$

and thus $B_i(y) - B_1(y)$ is constant on K_i. Therefore, K_i is nowhere dense for every $i = 2, \dots, N$. Consequently, the set $\bigcup_{i=2}^N K_i$ is also nowhere dense, and $K' = K_0 \setminus \bigcup_{i=2}^N K_i$ is not. Note that $\delta_k + \beta_{j_0}(y) \neq B_i(y)$ for every $y \in K'$ and $i = 2, \dots, N$.

Let $y \in K'$. The product on the right hand side of (16) contains the term $\gamma_k a_{j_0}(y)\, e^{(\delta_k + \beta_{j_0}(y))z}$. Now $\delta_k + \beta_{j_0}(y) > \delta_1 + \beta_1(y) = B_1(y)$ and $\delta_k + \beta_{j_0}(y) \neq B_i(y)$ for every $i \geq 2$ by $y \in K'$. Thus $\delta_k + \beta_{j_0}(y) \neq B_i(y)$ for every i and, consequently, this term must be cancelled out by other terms. That is, there are indices $i(y) < k$ and $j(y) > j_0$ such that $\delta_k + \beta_{j_0}(y) = \delta_{i(y)} + \beta_{j(y)}$. Now there must exist indices $i < k$ and $j > j_0$ and a non-nowhere dense subset K'' of K' such that $i(y) = i$ and $j(y) = j$ for every $y \in K''$. Then

$$\beta_j(y) - \beta_1(y) = \big(\beta_{j_0}(y) - \beta_1(y)\big) + (\delta_k - \delta_i)$$

for every $y \in K''$. Now $\beta_{j_0} - \beta_1$ is constant on K'' (even on K_0), and thus so is $\beta_j - \beta_1$. Therefore, $j \in J$. This, however, contradicts the fact that j_0 was the maximal element of J. This contradiction proves the finiteness of $\mathrm{sp}(L)$ in the case when (i) holds.

Next assume (ii). Then there is a nonzero complex number γ such that

$$E_y(z) = \big(e^{r(y)\gamma z} - c(y)\big) \cdot \sum_{j=1}^{m(y)} a_j(y)\, e^{\beta_j(y)z} \tag{17}$$

for every $y \in Y^n$, where $r(y) \neq 0$ is rational, $c(y), a_1(y), \dots, a_{m(y)}(y)$ are nonzero and $\beta_1(y), \dots, \beta_{m(y)}(y)$ are distinct for every y. We can prove, in the same way as in the case (i), that the numbers γ and $\beta_1(y), \dots, \beta_{m(y)}$ are real for every y. Since Y^n is a Baire space, there is a nonzero rational number r and there is a positive integer m such that the set $R = \{y \in U : r(y) = r,\ m(y) = m\}$ is not nowhere dense. Then there is a non-nowhere dense subset R_0 of R such that the ordering of the numbers $\beta_1(y), \dots, \beta_m(y)$ is the same for every $y \in R_0$ We may assume that $\beta_1(y) < \dots < \beta_m(y)$ $(y \in R_0)$. From this point we can arrive at a contradiction in the same way as in the case of (i), using (17) instead of (16).

This proves that sp(L) is finite. If $e^{\lambda_1 z}, \dots, e^{\lambda_s z}$ are the only exponential functions contained in L, then every exponential polynomial contained in L must be of the form $\sum_{i=1}^{s} p_i(z)\, e^{\lambda_i z}$, where $p_1, \dots, p_s$ are polynomials. Since the set of all polynomials is dense in $C(\mathbf{R})$ and $L \neq C(\mathbf{R})$, it follows that the degrees of $p_1, \dots, p_s$ must be bounded. As the set of exponential polynomials is dense in L, we find that each element of L is an exponential polynomial, which completes the proof of Theorem 2 in the case when $h \equiv 0$.

The general case can be reduced to the previous one in the same way as in the proof of Theorem 1. Again, it is enough to show that f_n is an exponential polynomial. Since $\Delta_b f_n$ satisfies the homogeneous version of (1), it follows that $\Delta_b f_n$ is an exponential polynomial for every b. Therefore, by Carroll's theorem [2], f_n is also an exponential polynomial. $\square$

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