3. Ideal non-symmetric solutions of the Tarry-Escott problem of degree five

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.04.2024**

We may apply the theorem of Gloden [2, p. 24] to the three-parameter ideal non-symmetric solution obtained above to derive a solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4, 6, \tag{2.16}$$

in terms of polynomials of degree six in three parameters. We, however, restrict ourselves to applying this theorem to the simpler solution (2.15), and obtain the following solution of the system of equations (2.16):

$$\begin{aligned} a_1 &= 9p^3 + 5p^2 - 3p + 5, & b_1 &= -6p^3 - 10p^2 - 8p, \\ a_2 &= -6p^3 - 15p^2 + 2p - 5, & b_2 &= -6p^3 + 15p^2 + 2p + 5, \\ a_3 &= -6p^3 + 10p^2 - 8p, & b_3 &= 9p^3 - 5p^2 - 3p - 5, \\ a_4 &= 9p^3 - 5p^2 + 7p + 5, & b_4 &= 9p^3 + 5p^2 + 7p - 5, \\ a_5 &= -6p^3 + 5p^2 + 2p - 5, & b_5 &= -6p^3 - 5p^2 + 2p + 5. \end{aligned} \tag{2.17}$$

When $p = -2$, this leads to the following solution of the system of equations (2.16):

$$(-101)^r + (-41)^r + (-21)^r + 59^r + 104^r = (-91)^r + (-71)^r + 24^r + 29^r + 109^r$$

where $r = 1, 2, 3, 4, 6$.

We note that additional parametric non-symmetric solutions of the Tarry-Escott problem of degree four may be obtained by taking a_i, b_i, as in (2.6), and instead of imposing the condition $a_6 = b_6$, we reduce one term on either side by solving (2.8) together with another condition such as $a_4 = b_6$ or $a_5 = b_6$. Solutions obtained in this manner are of degrees $6, 7$ or 8 in terms of three parameters.

3. IDEAL NON-SYMMETRIC SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE FIVE

To obtain ideal non-symmetric solutions of the Tarry-Escott problem of degree five, we have to obtain a solution of the system of equations

$$\sum_{i=1}^{6} a_i^r = \sum_{i=1}^{6} b_i^r, \qquad r = 1, 2, 3, 4, 5. \tag{3.1}$$

We will choose $a_1, a_2, a_3, a_4, a_5, a_6$ and $b_1, b_2, b_3, b_4, b_5, b_6$ as in (2.6) when (3.1) holds identically for $r = 1, 2, 4$. For $r = 3$, equation (3.1) reduces to (2.8) while for $r = 5$ it reduces to the equation:

$$\begin{aligned}(3.2)\quad & m_1n_1(m_1+n_1)x_1y_1(x_1+y_1)(m_1^2+m_1n_1+n_1^2)(x_1^2+x_1y_1+y_1^2)\\ &= m_2n_2(m_2+n_2)x_2y_2(x_2+y_2)(m_2^2+m_2n_2+n_2^2)(x_2^2+x_2y_2+y_2^2).\end{aligned}$$

It therefore suffices to solve equation (2.8) together with the following equation:

$$(3.3)\quad (m_1^2+m_1n_1+n_1^2)(x_1^2+x_1y_1+y_1^2) = (m_2^2+m_2n_2+n_2^2)(x_2^2+x_2y_2+y_2^2).$$

We take x_1, y_1, m_1, m_2, n_2 such that

$$(3.4)\qquad \begin{aligned} x_1 &= (t^2+t-1)x_2,\\ y_1 &= (t+1)^2y_2,\\ m_1 &= tx_2+ty_2,\\ m_2 &= (t^2+t-1)x_2+(t+1)^2y_2,\\ n_2 &= (-t^2-t)n_1.\end{aligned}$$

Substituting these values of x_1, y_1, m_1, m_2, n_2 in (2.8) and solving for n_1, we get

$$(3.5)\qquad n_1 = -\frac{(t^4+2t^3+t^2-1)x_2+(t^4+2t^3+t^2+t+1)y_2}{t^3+t^2-t-1},$$

and now (3.4) gives

$$(3.6)\qquad n_2 = \frac{(t^5+2t^4+t^3-t)x_2+(t^5+2t^4+t^3+t^2+t)y_2}{t^2-1}.$$

On substituting the values of n_1, n_2 given by (3.5) and (3.6), and the values of x_1, y_1, m_1, m_2 given by (3.4) in equation (3.3), we get the equation:

$$\begin{aligned}&(tx_2+(t+1)y_2)((t^2+t-1)x_2+(t+1)y_2)((t^2+t-1)x_2+(t^2+t)y_2)\\ &\quad\times (t+2)((2t^5+5t^4+3t^3-t^2-t+1)x_2+(t^5-6t^3-8t^2-4t-1)y_2)=0.\end{aligned}$$

Equating any of the first four factors on the left-hand side of this equation to zero leads either to trivial solutions or to known symmetric solutions of the Tarry-Escott problem of degree five. However, on equating the last factor to zero, we get

$$(3.7)\qquad \begin{aligned} x_2 &= t^5-6t^3-8t^2-4t-1,\\ y_2 &= -(2t^5+5t^4+3t^3-t^2-t+1),\end{aligned}$$

and now, using the relations (3.4), (3.5) and (3.6), we get the values of $x_1, y_1, m_1, m_2, n_1, n_2$. The values of $x_1, y_1, x_2, y_2, m_1, m_2, n_1, n_2$, may now be substituted in (2.6) to get a non-symmetric solution of the Tarry-Escott problem of degree five. After removing the common factors, this solution may be written as follows:

$$
\begin{aligned}
a_1 &= -3t^{11} - 20t^{10} - 58t^9 - 94t^8 - 106t^7 - 100t^6 - 40t^5 \\
&\qquad + 50t^4 + 50t^3 + 2t^2 - 5t\,, \\
a_2 &= 3t^{11} + 16t^{10} + 38t^9 + 71t^8 + 128t^7 + 149t^6 + 56t^5 \\
&\qquad - 37t^4 - 22t^3 + 5t^2 + t - 3\,, \\
a_3 &= 4t^{10} + 20t^9 + 23t^8 - 22t^7 - 49t^6 - 16t^5 \\
&\qquad - 13t^4 - 28t^3 - 7t^2 + 4t + 3\,, \\
a_4 &= 8t^{10} + 52t^9 + 127t^8 + 148t^7 + 85t^6 + 22t^5 \\
&\qquad + t^4 - 14t^3 - 23t^2 - 4t + 3\,, \\
a_5 &= 3t^{11} + 14t^{10} + 16t^9 - 29t^8 - 98t^7 - 89t^6 + 4t^5 + 55t^4 \\
&\qquad + 40t^3 + 13t^2 - 7t - 3\,, \\
a_6 &= -3t^{11} - 22t^{10} - 68t^9 - 98t^8 - 50t^7 + 4t^6 - 26t^5 \\
&\qquad - 56t^4 - 26t^3 + 10t^2 + 11t\,, \\
b_1 &= 2t^{10} + 13t^9 + 58t^8 + 151t^7 + 190t^6 + 79t^5 \\
&\qquad - 44t^4 - 41t^3 - 2t^2 - t\,, \\
b_2 &= -3t^{11} - 19t^{10} - 56t^9 - 116t^8 - 164t^7 - 104t^6 + 34t^5 \\
&\qquad + 76t^4 + 28t^3 + 4t^2 - t - 3\,, \\
b_3 &= 3t^{11} + 17t^{10} + 43t^9 + 58t^8 + 13t^7 - 86t^6 - 113t^5 \\
&\qquad - 32t^4 + 13t^3 - 2t^2 + 2t + 3\,, \\
b_4 &= 10t^{10} + 59t^9 + 140t^8 + 167t^7 + 98t^6 - t^5 \\
&\qquad - 58t^4 - 31t^3 + 14t^2 + 7t\,, \\
b_5 &= -3t^{11} - 23t^{10} - 67t^9 - 88t^8 - 25t^7 + 68t^6 + 71t^5 \\
&\qquad + 14t^4 - 13t^3 - 16t^2 - 2t + 3\,, \\
b_6 &= 3t^{11} + 13t^{10} + 8t^9 - 52t^8 - 142t^7 - 166t^6 - 70t^5 \\
&\qquad + 44t^4 + 44t^3 + 2t^2 - 5t - 3\,.
\end{aligned}
\tag{3.8}
$$

Here t is an arbitrary rational parameter and integer solutions of (3.1) are obtained by multiplying any rational numerical solution by a suitable constant. The a_i, b_i obtained above satisfy the relation $\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} b_i = 0$.

It therefore follows from the theorem of Gloden [2, p. 24] that these a_i, b_i must also satisfy the relation

$$\sum_{i=1}^{6} a_i^7 = \sum_{i=1}^{6} b_i^7 .$$

This is also verified by direct computation. Hence the a_i, b_i given by (3.8) constitute a solution of the following system of equations:

$$\sum_{i=1}^{6} a_i^r = \sum_{i=1}^{6} b_i^r , \qquad r = 1,2,3,4,5,7 .$$

As a numerical example, when $t = -3$, we get, after removal of common factors and suitable re-arrangement, the following solution

$$\begin{aligned}(-19323)^r + (-18689)^r + 3117^r + 5111^r + 14212^r + 15572^r \\ = (-20023)^r + (-17828)^r + 1017^r + 9787^r + 10236^r + 16811^r\end{aligned}$$

where $r = 1,2,3,4,5,7$.

REFERENCES

[1] CHOUDHRY, A. Ideal Solutions of the Tarry-Escott problem of degree four and a related diophantine system. *L'Enseignement Math. (2) 46* (2000), 313–323.

[2] GLODEN, A. *Mehrgradige Gleichungen.* Noordhoff, Groningen, 1944.

(Reçu le 27 novembre 2002)

Ajai Choudhry

High Commissioner
High Commission of India
P.O. Box 439, Mail Processing Centre
Airport Lama, Berakas
Bandar Seri Begawan, BB3577
Brunei
e-mail: ajaic203@yahoo.com