5. Splitting of the extension ASSOCIATED TO A NONCONNECTED COMPACT LIE GROUP

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5. Splitting of the extension associated to a nonconnected compact Lie group

Let G still denote a compact Lie group with a nonabelian connected component; we also assume the other notations introduced previously. In this final section, we make a few observations on the following problem: when is the natural extension associated to G split, i.e. when is G isomorphic to a semidirect product $G_0 \rtimes \Gamma$? Our aim is to relate this problem to the rest of this work. For a deeper analysis one should consult Chapter 6 in the book by Hofmann and Morris [14]. We start with a structure theorem for compact Lie groups based on centralizers of principal subgroups, similar to the "Sandwich Theorem for compact Lie groups" (see [14], Corollary 6.75, p. 272). This theorem shows that any compact Lie group is "sandwiched" in between two semidirect products closely related to it. We then recall a theorem of de Siebenthal and compare, in some particular cases, the question of the splitting of the extension associated to G to that of the extensions associated to the normalizer N of a maximal torus and to the centralizer Q of a principal diagonal, both introduced in Section 2. As an application of the "Sandwich" Theorem, the final proposition presents a "minimal" compact Lie group G such that the associated extension is *not* split.

Let $\overline{G}_0$ denote the adjoint group G_0/Z_0 ; it is well-known that the center of $\overline{G}_0$ is trivial.

THEOREM 5.1. *Let G be a compact Lie group with a nonabelian connected component. Then there exist two surjective homomorphisms*

$$\begin{array}{ccccc} G_s = G_0 \rtimes Z & \xrightarrow{\pi_1} & G & \xrightarrow{\pi_2} & \overline{G} = G/Z_0 \cong \overline{G}_0 \rtimes \Gamma, \\ (g_0, z) & \longmapsto & g_0 \cdot z & & \end{array}$$

where the centralizer Z of a fixed principal subgroup acts on G_0 by conjugation, and where π_2 is the canonical projection corresponding to the normal subgroup Z_0 of G.

For the kernels, we have $\ker \pi_1 \cong Z_0$ *and* $\ker \pi_2 = Z_0$ *; in particular*

$$G \cong (G_0 \rtimes Z)/Z_0 \qquad \textit{and} \qquad G/Z_0 \cong \overline{G}_0 \rtimes \Gamma \,.$$

Proof. As $\overline{G}_0$ is centerless, $\overline{G}$ must be isomorphic to $\overline{G}_0 \rtimes \Gamma$ by the proof of Proposition 3.1. The other assertions about π_2 are clear.

The map π_1 is well-defined and surjective (because Z intersects every component of G). Straightforward computations show that it is a homomorphism and that $\ker \pi_1 \cong Z_0$. $\square$

REMARK 5.2. 1) The component of the identity of G_s is equal to G_o if and only if G_o is semisimple.

2) The present version of the "Sandwich" Theorem has the advantage of being more explicit than the one in [14] (the result therein is an existence theorem). Its drawbacks are the fact that Z is not finite if G_o is not semisimple, and that it obviously makes no sense for compact Lie groups with an abelian connected component.

3) Given a homomorphism $\varphi\colon \Gamma \to \mathrm{Out}(G_o)$, and an extension $Z_o \overset{\mu}{\hookrightarrow} E \overset{\nu}{\twoheadrightarrow} \Gamma$ for which the action coincides with the "restriction" $\bar{\varphi}$, there is a more direct way than the cohomological one to recover the corresponding compact Lie group G, i.e. the one that fits into the commutative diagram

$$\begin{tikzcd} Z_o \arrow[r, hook] \arrow[d, hook] & E \arrow[r, two heads] \arrow[d, hook] & \Gamma \arrow[d, equal] \\ G_o \arrow[r, hook] & G \arrow[r, two heads] & \Gamma \end{tikzcd}$$

Let us define the composition $\bar{\sigma}\colon E \overset{\nu}{\twoheadrightarrow} \Gamma \overset{\varphi}{\to} \mathrm{Out}(G_o) \overset{s}{\to} \mathrm{Aut}(G_o)$, where s is as in Theorem 2.4. Then by Bourbaki (see [4], Lemme 7, pp. 210–211), we have

$$G = (G_o \rtimes_{\bar{\sigma}} E)/\Delta Z_o\,,$$

where ΔZ_o is the image of the injection $z_o \mapsto (z_o^{-1}, \mu(z_o))$. Taking $E = Z$, this gives another proof of the assertions concerning π_1 in Theorem 5.1.

Using Cartan subgroups (in the sense of Segal [22], i.e. those Adams called "SS subgroups" in honour of Segal and de Siebenthal [1]), de Siebenthal gave some explicit sufficient conditions for the splitting of the extension associated to G ([9], Théorème p. 74).

THEOREM 5.3 (de Siebenthal). *Let G be a compact Lie group with G_o simply connected, or of adjoint type (i.e. Z_o is trivial). If $\Gamma = \pi_0(G)$ is cyclic then G is a semidirect product, i.e. $G \cong G_o \rtimes \Gamma$.*

A relationship with the splitting of the extension associated to the normalizer of a maximal torus N in G is given in the next proposition.

PROPOSITION 5.4. *If the group of components Γ of G is nilpotent, then the extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ is split if and only if the extension $N_o \hookrightarrow N \twoheadrightarrow \Gamma$ is split.*

Proof. The "if" part is clear. Conversely, let $s\colon \Gamma \to G$ be a section. By a result in Bourbaki (see [5], Corollaire 4, p. 49), any nilpotent subgroup of a compact Lie group is contained in the normalizer of some maximal torus. Therefore, if needed after conjugation by an element in G_o, we have $s(\Gamma) \subset N$, and we can conclude that the extension associated to N is split. □

For an extended maximal torus Q, the extensions can be related as follows.

PROPOSITION 5.5. *If the group of components* Γ *of* G *is cyclic, then the extension* $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ *is split if and only if the extension* $T \hookrightarrow Q \twoheadrightarrow \Gamma$ *is split.*

Proof. The proposition readily follows from the fact that the conjugates of Q cover G. □

REMARK 5.6. This latter proposition fails in general. An obstruction to the splitting of the extension associated to the extended maximal torus Q can be found in a paper by Oliver; this obstruction involves the representation ring of G and its relation with the family of all p-toral subgroups of G (see [19], Corollary 3.11). In particular, Oliver constructs a compact Lie group $G = \mathrm{SU}(2) \rtimes (\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/3)$ such that the extension corresponding to the extended maximal torus Q is *not* split ([19], pp. 376–377).

We conclude with the promised example.

PROPOSITION 5.7. *Let* $\mathrm{D}_8 = \langle r, s \mid r^4 = s^2 = e,\ srs^{-1} = r^{-1} \rangle$ *be a presentation of the dihedral group. Then the quotient*

$$G = (\mathrm{SU}(2) \times \mathrm{D}_8)/\Delta\mathbf{Z}/2\,,$$

where $\Delta\mathbf{Z}/2$ *denotes the central subgroup generated by* $(-\mathbf{1}, r^2)$, *is a compact Lie group with* $G_o \cong \mathrm{SU}(2)$ *and* $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$, *the* 4*-group of Klein. The associated extension*

$$\mathrm{SU}(2) \hookrightarrow G \twoheadrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2$$

is not split. Among the extensions associated to a compact Lie group with a nonabelian connected component, it is a minimal one having the property of being non-split, in the sense that the rank of the connected component and the size of the group of components are minimal. Moreover replacing the *connected component by* SO(3) *(i.e. by "any" group of the same rank), or the group of components by* $\mathbf{Z}/4$ *(i.e. by "any" group of the same size), will force the extension to be split.*

Proof. The assertions about the connected component and the group of components are clear. Let us show that the extension associated to G is not split. Let us denote $[g,\gamma] \in G$ the image of $(g,\gamma) \in \mathrm{SU}(2) \times \mathrm{D}_8$ under the canonical projection. Let $\mathbf{S}^1$ denote the standard maximal torus in SU(2), and let N denote its normalizer in G. We have

$$N = \left\{[t,e] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,e] : t \in \mathbf{S}^1\right\} \amalg \left\{[t,r] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,r] : t \in \mathbf{S}^1\right\}$$
$$\amalg \left\{[t,s] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,s] : t \in \mathbf{S}^1\right\} \amalg \left\{[t,rs] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,rs] : t \in \mathbf{S}^1\right\}.$$

By contradiction, suppose that the extension associated to G is split, i.e. there exists a section. As $\mathbf{Z}/2 \times \mathbf{Z}/2$ is abelian, thus nilpotent, we deduce, by Proposition 5.4, that the extension associated to N is also split. We want to show that this is not possible by considering the elements of order 2 in N. For $n = 0, 1$, a straightforward calculation shows that in the component corresponding to $r^n s$, an element $[t, r^n s]$ is of order 2 if and only if $t = \pm\mathbf{1}$, and that the sub-component $\left\{[jt, r^n s] : t \in \mathbf{S}^1\right\}$ does not contain any element of order 2. Two of the three non-trivial elements in $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ must thus be mapped by the section to $[\pm\mathbf{1}, s]$ and $[\pm\mathbf{1}, rs]$. Therefore, as the section is a homomorphism, the image of the third non-trivial element is

$$[\pm\mathbf{1}, rs] \cdot [\pm\mathbf{1}, s] = [\pm\mathbf{1}, r] \ ,$$

which is not of order 2. A contradiction that shows that the extension associated to G is not split.

The property of minimality follows by Theorem 5.3, and by the fact that any extension with SO(3) as normal subgroup is a direct product (because SO(3) is complete, i.e. centerless and with trivial outer automorphism group). □

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