3.6 Category O

DEFINITION 3.13. The *spherical subalgebra* of H_c is the algebra eH_ce.

Notice that $1 \notin eH_ce$. On the other hand, since $ex = xe = e$ for $x \in eH_ce$, e is the unit for the spherical subalgebra. We can embed both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ in the spherical subalgebra as follows. Take $f \in \mathbf{C}[\mathfrak{h}^*]^W$ (the other case is identical) and set $m_e(f) = fe$. Since f is invariant, we have $efe = fe^2 = fe = m_e(f)$, so that m_e actually maps $\mathbf{C}[\mathfrak{h}^*]^W$ to eH_ce. The injectivity is clear from the PBW-theorem. As for the fact that m_e is a homomorphism, we have $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$. From now on, we will consider both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ as subalgebras of the spherical subalgebra.

3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras H_c and eH_ce.

DEFINITION 3.14. The category $\mathcal{O}(H_c)$ (resp. $\mathcal{O}(eH_ce)$) is the full subcategory of the category of H_c-modules (resp. eH_ce-modules) whose objects are the modules M such that

1) M is finitely generated.
2) For all $v \in M$, the subspace $\mathbf{C}[\mathfrak{h}^*]^W v \subset M$ is finite dimensional.

We can define a functor

$$F\colon \mathcal{O}(H_c) \to \mathcal{O}(eH_ce)$$

by setting $F(M) = eM$. It is easy to show that $F(M)$ is an object of $\mathcal{O}(eH_ce)$.

We are now going to explain how to construct some modules in $\mathcal{O}(H_c)$ which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take $\lambda \in \mathfrak{h}^*$. Denote by $W_\lambda \subset W$ the stabilizer of λ. Take an irreducible W_λ-module τ. We define a structure of $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$-module on τ by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau,\ w \in W_\lambda,\ f \in \mathbf{C}[\mathfrak{h}^*].$$

It is easy to see that this action is well defined and we denote this module by $\lambda\#\tau$. We can then consider the H_c-module

$$M(\lambda,\tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*]\rtimes\mathbf{C}[W_\lambda]} \lambda\#\tau .$$

This is called a Whittaker module. In the special case $\lambda = 0$ (and hence $W_\lambda = W$), the module $M(0,\tau)$ is called a Verma module. It is clear that these are objects of $\mathcal{O}$. Notice that as $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(\lambda,\tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$.

EXAMPLE 3.15. If $\lambda = 0$ and $\tau = \mathbf{1}$ is the trivial representation of W, the Verma module $M(0,\mathbf{1}) = \mathbf{C}[\mathfrak{h}]$. The action of $\mathbf{C}[\mathfrak{h}]$ is given by multiplication, that of $\mathbf{C}[\mathfrak{h}^*]$ is generated by the Dunkl operators and W acts in the usual way.

3.7 GENERIC c

Opdam and Rouquier have recently studied the structure of the categories $\mathcal{O}(H_c)$, $\mathcal{O}(eH_ce)$, and found that it is especially simple if c is "generic" in a certain sense. Namely, recall that for a W-invariant function $q\colon \Sigma \to \mathbf{C}^*$ one can define the *Hecke algebra* $\mathrm{He}_q(W)$ to be the quotient of the group algebra of the fundamental group of U/W by the relations $(T_s-1)(T_s+q_s) = 0$, where T_s is the image in U/W of a small half-circle around the hyperplane of s in the counterclockwise direction. It is well known that $\mathrm{He}_q(W)$ is an algebra of dimension $|W|$, which coincides with $\mathbf{C}[W]$ if $q = 1$. It is also known that $\mathrm{He}_q(W)$ is semisimple (and isomorphic to $\mathbf{C}[W]$ as an algebra) unless q_s belongs for some s to a finite set of roots of unity depending on W (see [Hu]).

DEFINITION 3.16. The function c is said to be *generic* if for $q = e^{2\pi i c}$, the Hecke algebra $\mathrm{He}_q(W)$ is semisimple.

In particular, any irrational c is generic, and (more important for us) an integer valued c is generic (since in this case $q = 1$). We can now state the following central result:

THEOREM 3.17 (Opdam-Rouquier [OR]; see also [BEG] for an exposition). *If c is generic (in particular, if c takes non negative integer values), then the irreducible objects in $\mathcal{O}$ are exactly the modules $M(\lambda,\tau)$. Moreover, the category $\mathcal{O}$ is semisimple.*

We also have

THEOREM 3.18 ([OR]). *If c is generic then the functor F is an equivalence of categories.*

From Theorem 3.17 we can deduce

THEOREM 3.19 ([BEG]). *If c is generic, then H_c is a simple algebra.*

In the case $c = 0$, we get the simplicity of $\mathbf{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathbf{C}[W]$, which is well known.