# 3. Lecture 3

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The solution can easily be computed by differentiating the first equation and then subtracting the second, thus obtaining the new system

$$\begin{cases} \psi'' - \frac{2}{x}\psi' = k^2\psi\,, \\ \psi'' - (\frac{1}{x} + k^2x)\psi' = -k^3x\psi\,. \end{cases}$$

Taking the difference, we get the first order equation

$$\psi' = \frac{k^2x}{kx-1}\psi\,,$$

whose solution (up to constants) is given by $\psi = (kx-1)\,e^{kx}$.

In fact, one can easily calculate $\psi_m$ for a general $m$.

PROPOSITION 2.12. $\psi_m(k,x) = (x\partial - 2m+1)(x\partial - 2m-1)\cdots(x\partial - 1)\,e^{kx}$.

*Proof.* We could use the direct method of Example 2.11, but it is more convenient to proceed differently. Namely, we have

$$(\partial^2 - \frac{2m}{x}\partial)(x\partial - 2m+1) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)$$

as it is easy to verify directly. So using induction on $m$ starting with $m = 0$, we get

$$(\partial^2 - \frac{2m}{x}\partial)\psi_m(k,x) = (x\partial - 2m+1)(\partial^2 - \frac{2(m-1)}{x}\partial)\psi_{m-1}(k,x) = k^2\psi_m(k,x)\,,$$

and $\psi_m(k,x)$ is our solution. $\square$

## 3. LECTURE 3

### 3.1 SHIFT OPERATOR AND CONSTRUCTION OF THE BAKER-AKHIEZER FUNCTION

In Lecture 2, we have introduced the Baker-Akhiezer function $\psi(k,x)$ for the operator

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)}\partial_{\alpha_s}\,.$$

The way to construct $\psi(k,x)$ is via the Opdam shift operator. Given a function $m\colon \Sigma \to \mathbf{Z}_+$, Opdam showed in [Op1] that there exists a unique $W$-invariant differential operator $S_m$ of the form $\delta_m(x)\delta_m(\partial_x)+l.o.t.$, with $\delta_m(x)=\prod_{s\in\Sigma}\alpha_s^{m_s}$ such that

$$L_q S_m = S_m q(\partial)$$

for every $q\in\mathbf{C}[\mathfrak{h}]=\mathbf{C}[q_1,\dots,q_n]$. From this, if we set $\psi(k,x)=S_m e^{(k,x)}$, we get

$$L_q\psi = S_m q(\partial)\, e^{(k,x)} = q(k)\psi\,, \tag{7}$$

$q\in\mathbf{C}[q_1,\dots,q_n]$.

We claim that equation (7) must in fact hold for all $q\in Q_m$. Indeed, near a generic point $x$, the functions $\psi(wk,x)$ are obviously linearly independent and satisfy (7) for symmetric $q$. Thus, they are a basis in the space of solutions (we know that this space is $|W|$-dimensional). Consider the matrix of $L_q$ in this basis for any $q\in Q_m$. Since $\psi(k,x)$ is a polynomial multiplied by $e^{(k,x)}$, this matrix must be diagonal with eigenvalues $q(k)$, as desired.

EXAMPLE 3.1. As we have seen in the previous section, for $W=\mathbf{Z}/2$ and $\mathfrak{h}=\mathbf{C}$,

$$S_m=(x\partial-2m+1)(x\partial-2m-1)\cdots(x\partial-1)\,.$$

### 3.2 BEREST'S FORMULA FOR $L_q$

We are now going to give an explicit construction of the operators $L_q$ for any $q\in Q_m$.

Let us identify, using our $W$-invariant scalar product, $\mathfrak{h}$ with $\mathfrak{h}^*$, and let us choose a orthonormal basis $x_1,\dots,x_n$ in $\mathfrak{h}^*$. If $x\in\mathfrak{h}^*$, we will write $D_x$ for the Dunkl operator relative to the vector in $\mathfrak{h}$ corresponding to $x$ under our identification. Thus

$$L=\sum_{i=1}^n D_{x_i}^2\,.$$

PROPOSITION 3.2 (Berest [Be]). *If $q\in Q_m$ is a homogeneous element of degree $d$, then*

$$(\operatorname{ad}L)^{d+1}q=0\,.$$

*Proof.* It is enough to prove that

$$((\operatorname{ad}L)^{d+1}q)\psi(k,x)=0\,.$$

Indeed, it follows from the definition of $\psi(k,x)$ that in the ring $\mathcal{D}(U)$ this implies: $((\operatorname{ad}L)^{d+1}q)\,S_m=0$, so that $(\operatorname{ad}L)^{d+1}\,q=0$, since $\mathcal{D}(U)$ is a domain.

Given $q \in Q_m$, we will denote by $L_q^{(k)}$ the operator $q(D_{k_1}, \dots, D_{k_n})$. Notice that since $\psi(k,x) = \psi(x,k)$, we have $L_q^{(k)}\psi = q(x)\psi$. Thus we deduce, for $p, q, r \in Q_m$,

$$\begin{aligned} L_q r(x) L_p \psi &= L_q r(x) p(k) \psi = p(k) L_q r(x)\, \psi \\ &= p(k) L_q L_r^{(k)} \psi = p(k) L_r^{(k)} L_q \psi = p(k) L_r^{(k)} q(k)\, \psi\,. \end{aligned}$$

It follows that

$$(\operatorname{ad} L)^{d+1} q \psi = (-1)^{d+1} (\operatorname{ad}(\sum_{i=1}^n k_i^2))^{d+1} L_q^{(k)} \psi\,.$$

Since $L_q$ is a differential operator of degree $d$, we get $\operatorname{ad}(\sum_{i=1}^n k_i^2)^{d+1} L_q^{(k)} = 0$, as desired. $\square$

Notice now that the operator $(adL)^d q(x)$ commutes with $L$. Its symbol is given by $(\operatorname{ad}\Delta)^d q(x) = 2^d d! q(\partial)$. So we deduce the following

COROLLARY 3.3 (Berest's formula, [Be]). *If $q \in Q_m$ is homogeneous of degree $d$, then*

$$L_q = \frac{1}{2^d d!} (\operatorname{ad} L)^d q(x)\,.$$

*Proof.* This is clear from Proposition 2.8, once we remark that $(\operatorname{ad} L)^d q(x)$ has the required homogeneity. $\square$

We want to give a representation theoretical interpretation of what we have just seen. Consider the three operators

$$F = \frac{\sum_{i=1}^n x_i^2}{2}\,, \quad E = -\frac{L}{2}\,, \quad H = [E, F]\,. \tag{8}$$

It is easy to check that $[H, E] = 2E$, $[H, F] = -2F$. We deduce that the elements $E, F, H$ span an $\mathfrak{sl}(2)$ Lie subalgebra of $\mathcal{D}(U)$. Thus $\mathfrak{sl}(2)$ acts by conjugation on $\mathcal{D}(U)$. We can then reformulate Proposition 3.2 as follows:

PROPOSITION 3.4. *Any polynomial $q \in Q_m$ of degree $d$ is a lowest weight vector for the $\mathfrak{sl}(2)$-action of weight $-d$ and generates a finite dimensional module (necessarily of dimension $d+1$) for which $L_q$ is a highest weight vector.*

*Proof.* An easy direct computation shows that

$$H = [E, F] = -\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + C,$$

where $C$ is a constant. Thus if $q$ is homogeneous of degree $d$, we have $[H, L_q] = dL_q$.

This and the fact that $[L, L_q] = 0$, implies that $L_q$ is a highest weight vector of weight $d$. Also since $F$ is a polynomial, we deduce that $\operatorname{ad} F^{d+1} L_q = 0$, so that $L_q$ generates a $(d+1)$-dimensional irreducible $\mathfrak{sl}(2)$-module. $\square$

One last property about these operators is given by

PROPOSITION 3.5 ([FV]). *For any $q \in Q_m$, the operator $L_q$ preserves $Q_m$.*

*Proof.* Let us begin by proving that $L$ preserves $Q_m$.

Take $f \in Q_m$, so that for any $s \in \Sigma$, $f - {}^s f = \alpha_s^{2m_s+1} t$, $t \in \mathbf{C}[\mathfrak{h}]$. Let us start by showing that $Lf$ is a polynomial. Clearly $Lf = \delta_*^{-1} q$, with $q \in \mathbf{C}[\mathfrak{h}]$, and $\delta_* = \prod_{s: m_s \neq 0} \alpha_s$. Since $L$ is $W$-invariant, $Lf - {}^s(Lf) = L(f - {}^s f)$ is clearly divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In particular, it is always regular along the reflection hyperplane of $s$. On the other hand, since $Lf - {}^s(Lf) = \delta_*^{-1}(q + {}^s q)$, we deduce that $q + {}^s q$ is divisible by $\alpha_s$ if $m_s > 0$. But then $q = ((q + {}^s q) + (q - {}^s q))/2$ is divisible by $\alpha_s$ if $m_s > 0$, hence it is divisible by $\delta_*$, so that $Lf$ lies in $\mathbf{C}[\mathfrak{h}]$.

We have already remarked that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s-1}$ if $m_s > 0$. In fact

$$L(f - {}^s f) = (L\alpha_s^{2m_s+1})t + \alpha_s^{2m_s} \tilde{t},$$

where $\tilde{t}$ is a suitable polynomial.

But since

$$\begin{aligned} L\alpha_s^{2m_s+1} &= 2m_s(2m_s+1)(\alpha_s, \alpha_s)\alpha_s^{2m_s-1} - 2m_{s'}(2m_s+1)\sum_{s' \in \Sigma} (\alpha_{s'}, \alpha_s)\frac{\alpha_s^{2m_s}}{\alpha_{s'}} \\ &= -2m_{s'}(2m_s+1) \sum_{s' \in \Sigma, s' \neq s} (\alpha_{s'}, \alpha_s)\frac{{\alpha_s}^{2m}}{\alpha_{s'}}, \end{aligned}$$

we deduce that $L(f - {}^s f)$ is divisible by $\alpha_s^{2m_s}$. On the other hand, since $L(f - {}^s f) = Lf - {}^s(Lf)$, this polynomial is either zero or it must vanish to odd order on the reflection hyperplane of $s$. We deduce that it must be divisible by $\alpha_s^{2m_s+1}$, proving that $Lf \in Q_m$.

We now pass to a general $L_q$, $q \in Q_m$. We may assume that $q$ is homogeneous of, say, degree $d$. By Corollary 3.3 we have that $L_q$ is a non zero multiple of $(adL)^d(q)$. Since both $q$ and $L$ preserve $Q_m$, our claim follows. $\square$

### 3.3 DIFFERENTIAL OPERATORS ON $X_m$

Now let us return to the algebra of differential operators $\mathcal{D}(X_m)$. Notice that $\mathcal{D}(X_m)$ contains two commutative subalgebras (both isomorphic to $Q_m$). The first is $Q_m$ itself, the second is the subalgebra $Q_m^\dagger$ consisting of the differential operators of the form $L_q$ with $q \in Q_m$. It is possible to prove

THEOREM 3.6 ([BEG]). *$\mathcal{D}(X_m)$ is generated by $Q_m$ and $Q_m^\dagger$.*

Notice that by Corollary 3.3 we in fact have that $\mathcal{D}(X_m)$ is generated by $Q_m$ and by $L$.

EXAMPLE 3.7. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$ we get that $\mathcal{D}(X_m)$ is generated by the operators

$$x^2, \quad x^{2m+1}, \quad \frac{d^2}{dx^2} - \frac{2m}{x}\frac{d}{dx}.$$

Theorem 3.6 together with Proposition 3.4, imply

COROLLARY 3.8 ([BEG]). *$\mathcal{D}(X_m)$ is locally finite dimensional under the action of the Lie algebra $\mathfrak{sl}(2)$ defined in (8).*

This Corollary implies that our $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$ can be integrated to an action of the group $SL(2)$. In particular we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q = L_q$$

for all $q \in Q_m$. This transformation is a generalization of the Fourier transform, since it reduces to the usual Fourier transform on differential operators on $\mathfrak{h}$ when $m = 0$.

EXAMPLE 3.9. If $W = \mathbf{Z}/2$, $\mathfrak{h} = \mathbf{C}$, we get that the monomials $\{x^{2i}\} \cup \{x^{2i+2m+1}\}$ are (up to constants) all lowest weight vectors for the $\mathfrak{sl}(2)$ action on $\mathcal{D}(X_m)$. $x^n$ has weight $-n$. We deduce that $\mathcal{D}(X_m)$ is isomorphic as a $\mathfrak{sl}(2)$-module to the direct sum of the irreducible representations of dimension $n+1$ for $n$ even or $n = 2(m+i)+1$, each with multiplicity one.

### 3.4 The Cherednik algebra

Let us now return to the algebra $\mathcal{A}$ of operators on $U$ generated by $\mathcal{D}(U)$ and $W$. This algebra contains the Dunkl operators

$$D_y := \partial_y + \sum_{s\in\Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s}(s-1).$$

LEMMA 3.10. *The following relations hold:*

$$[x_i, x_j] = [D_{x_i}, D_{x_j}] = 0, \quad \forall 1 \le i,j \le n$$
$$[D_{x_i}, x_j] = \delta_{i,j} + \sum_{s\in\Sigma} c_s \frac{(x_i,\alpha_s)(x_j,\alpha_s)}{(\alpha_s,\alpha_s)} s, \quad \forall 1 \le i,j \le n$$
$$wxw^{-1} = w(x), \quad wD_y w^{-1} = D_{w(y)}, \quad \forall w \in W, \ x \in \mathfrak{h}^*, \ y \in \mathfrak{h}.$$

*Proof.* The proof is an easy computation, except for the relations $[D_{x_i}, D_{x_j}] = 0$, which follow from Theorem 2.6. □

This lemma motivates the following definition.

DEFINITION 3.11 (see e.g. [EG]). The *Cherednik algebra* $H_c$ is an associative algebra with generators $x_i, y_i, i = 1, \ldots, n$, and $w \in W$, with defining relations

$$[x_i, x_j] = [y_i, y_j] = 0, \quad \forall 1 \le i,j \le n$$
$$[y_i, x_j] = \delta_{i,j} + \sum_{s\in\Sigma} c_s \frac{(x_i,\alpha_s)(x_j,\alpha_s)}{(\alpha_s,\alpha_s)} s, \quad \forall 1 \le i,j \le n$$
$$wxw^{-1} = w(x), \ wyw^{-1} = w(y), \ w \cdot w' = ww', \ \forall w, w' \in W, \ x \in \mathfrak{h}^*, \ y \in \mathfrak{h}.$$

This algebra was introduced by Cherednik as a rational limit of his double affine Hecke algebra defined in [Ch]. Notice that if $c = 0$ then $H_0 = \mathcal{D}(\mathfrak{h}) \rtimes \mathbf{C}[W]$.

Lemma 3.10 implies that the algebra $H_c$ is equipped with a homomorphism $\phi\colon H_c \to \mathcal{A}$, given by $w \to w$, $x_i \to x_i$, $y_i \to D_{x_i}$.

Cherednik proved the following theorem.

THEOREM 3.12 (Poincaré-Birkhoff-Witt theorem). *The multiplication map*

$$\mu\colon \mathbf{C}[\mathfrak{h}] \otimes \mathbf{C}[\mathfrak{h}^*] \otimes \mathbf{C}[W] \to H_c$$

*given by* $\mu(f(x) \otimes g(y) \otimes w) = f(x)\,g(y)w$ *is an isomorphism of vector spaces.*

*Proof.* It is easy to see that the map $\mu$ is surjective. Thus, we only have to show that it is injective. In other words, we need to show that monomials $x_1^{i_1}\dots x_n^{i_n}y_1^{j_1}\dots y_n^{j_n}w$ are linearly independent in $H_c$. To do this, it suffices to show that the images of these monomials under the homomorphism $\phi$, i.e. $x_1^{i_1}\dots x_n^{i_n}D_{x_1}^{j_1}\dots D_{x_n}^{j_n}w$, are linearly independent.

Given an element $A\in\mathcal{A}$, writing $A=\sum_{w\in W}P_w w$ with $P_w\in\mathcal{D}(U)$ we define the order of $A$, ord$A$, as the maximum of the orders of the $P_w$'s. Notice that ord$AB\leq$ord$A$+ord$B$. We now remark that for any sequence of non negative indices $(i_1,\dots,i_n)$,

$$D_{x_1}^{i_1}\cdots D_{x_n}^{i_n}=\partial_{x_1}^{i_1}\cdots\partial_{x_n}^{i_n}+l.o.t.$$

Indeed this is true for $D_{x_i}$. We proceed by induction on $r=i_1+\dots+i_n$. We can clearly assume $i_1>0$, so by induction,

$$D_{x_1}^{i_1}\cdots D_{x_n}^{i_n}=(\partial_{x_1}+l.o.t.)(\partial_{x_1}^{i_1-1}\cdots\partial_{x_n}^{i_n}+l.o.t.)=\partial_{x_1}^{i_1}\cdots\partial_{x_n}^{i_n}+l.o.t.$$

From this we deduce that for any pair of multiindices $I=(i_1,\dots,i_n)$, $J=(j_1,\dots,j_n)$, $w\in W$, setting $x_I={x_1}^{i_1}\cdots{x_n}^{i_n}$, $D_J=D_{x_1}^{j_1}\cdots D_{x_n}^{j_n}$, $\partial_J=\partial_{x_1}^{j_1}\cdots\partial_{x_n}^{j_n}$, we have

$$x_ID_Jw=x_I\partial_Jw+l.o.t.$$

Using this and the linear independence of the elements $x_I\partial_Jw$, it is immediate to conclude that the elements $x_ID_Jw$ are linearly independent, proving our claim. $\square$

REMARK 1. We see that the homomorphism $\phi$ identifies $H_c$ with the subalgebra of $\mathcal{A}$ generated by $\mathbf{C}[\mathfrak{h}]$, the Dunkl operators $D_y$, $y\in\mathfrak{h}$ and $W$.

REMARK 2. Another way to state the PBW theorem is the following. Let $F^\bullet$ be a filtration on $H_c$ defined by $\deg(x_i)=\deg(y_i)=1$, $\deg(w)=0$. Then we have a natural surjective mapping from $\mathbf{C}[\mathfrak{h}\times\mathfrak{h}^*]\rtimes W$ to the associated graded algebra gr$(H_c)$. The PBW theorem claims that this map is in fact an isomorphism.

### 3.5 THE SPHERICAL SUBALGEBRA

Let us now introduce the idempotent

$$e=\frac{1}{W}\sum_{w\in W}w\in\mathbf{C}[W].$$

DEFINITION 3.13. The *spherical subalgebra* of $H_c$ is the algebra $eH_ce$.

Notice that $1 \notin eH_ce$. On the other hand, since $ex = xe = e$ for $x \in eH_ce$, $e$ is the unit for the spherical subalgebra. We can embed both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ in the spherical subalgebra as follows. Take $f \in \mathbf{C}[\mathfrak{h}^*]^W$ (the other case is identical) and set $m_e(f) = fe$. Since $f$ is invariant, we have $efe = fe^2 = fe = m_e(f)$, so that $m_e$ actually maps $\mathbf{C}[\mathfrak{h}^*]^W$ to $eH_ce$. The injectivity is clear from the PBW-theorem. As for the fact that $m_e$ is a homomorphism, we have $m_e(fg) = fge = fge^2 = fege = m_e(f)m_e(g)$. From now on, we will consider both $\mathbf{C}[\mathfrak{h}^*]^W$ and $\mathbf{C}[\mathfrak{h}]^W$ as subalgebras of the spherical subalgebra.

## 3.6 CATEGORY $\mathcal{O}$

We are now going to study representations of the algebras $H_c$ and $eH_ce$.

DEFINITION 3.14. The category $\mathcal{O}(H_c)$ (resp. $\mathcal{O}(eH_ce)$) is the full subcategory of the category of $H_c$-modules (resp. $eH_ce$-modules) whose objects are the modules $M$ such that

1) $M$ is finitely generated.
2) For all $v \in M$, the subspace $\mathbf{C}[\mathfrak{h}^*]^W v \subset M$ is finite dimensional.

We can define a functor

$$F\colon \mathcal{O}(H_c) \to \mathcal{O}(eH_ce)$$

by setting $F(M) = eM$. It is easy to show that $F(M)$ is an object of $\mathcal{O}(eH_ce)$.

We are now going to explain how to construct some modules in $\mathcal{O}(H_c)$ which, by analogy with the case of enveloping algebras of semisimple Lie algebras, we will call Whittaker and Verma modules. First, take $\lambda \in \mathfrak{h}^*$. Denote by $W_\lambda \subset W$ the stabilizer of $\lambda$. Take an irreducible $W_\lambda$-module $\tau$. We define a structure of $\mathbf{C}[\mathfrak{h}^*] \rtimes \mathbf{C}[W_\lambda]$-module on $\tau$ by

$$(fw)v = f(\lambda)(wv) \quad \forall v \in \tau\,,\ w \in W_\lambda\,,\ f \in \mathbf{C}[\mathfrak{h}^*]\,.$$

It is easy to see that this action is well defined and we denote this module by $\lambda\#\tau$. We can then consider the $H_c$-module

$$M(\lambda, \tau) = H_c \otimes_{\mathbf{C}[\mathfrak{h}^*]\rtimes\mathbf{C}[W_\lambda]} \lambda\#\tau\,.$$

This is called a Whittaker module. In the special case $\lambda = 0$ (and hence $W_\lambda = W$), the module $M(0,\tau)$ is called a Verma module. It is clear that these are objects of $\mathcal{O}$. Notice that as $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(\lambda,\tau) = \mathbf{C}[\mathfrak{h}] \otimes_{\mathbf{C}} \mathbf{C}[W] \otimes_{\mathbf{C}[W_\lambda]} \tau$.

EXAMPLE 3.15. If $\lambda = 0$ and $\tau = \mathbf{1}$ is the trivial representation of $W$, the Verma module $M(0, \mathbf{1}) = \mathbf{C}[\mathfrak{h}]$. The action of $\mathbf{C}[\mathfrak{h}]$ is given by multiplication, that of $\mathbf{C}[\mathfrak{h}^*]$ is generated by the Dunkl operators and $W$ acts in the usual way.

## 3.7 GENERIC $c$

Opdam and Rouquier have recently studied the structure of the categories $\mathcal{O}(H_c)$, $\mathcal{O}(eH_ce)$, and found that it is especially simple if $c$ is "generic" in a certain sense. Namely, recall that for a $W$-invariant function $q\colon \Sigma \to \mathbf{C}^*$ one can define the *Hecke algebra* $\mathrm{He}_q(W)$ to be the quotient of the group algebra of the fundamental group of $U/W$ by the relations $(T_s - 1)(T_s + q_s) = 0$, where $T_s$ is the image in $U/W$ of a small half-circle around the hyperplane of $s$ in the counterclockwise direction. It is well known that $\mathrm{He}_q(W)$ is an algebra of dimension $|W|$, which coincides with $\mathbf{C}[W]$ if $q = 1$. It is also known that $\mathrm{He}_q(W)$ is semisimple (and isomorphic to $\mathbf{C}[W]$ as an algebra) unless $q_s$ belongs for some $s$ to a finite set of roots of unity depending on $W$ (see [Hu]).

DEFINITION 3.16. The function $c$ is said to be *generic* if for $q = e^{2\pi i c}$, the Hecke algebra $\mathrm{He}_q(W)$ is semisimple.

In particular, any irrational $c$ is generic, and (more important for us) an integer valued $c$ is generic (since in this case $q = 1$). We can now state the following central result:

THEOREM 3.17 (Opdam-Rouquier [OR]; see also [BEG] for an exposition). *If $c$ is generic (in particular, if $c$ takes non negative integer values), then the irreducible objects in $\mathcal{O}$ are exactly the modules $M(\lambda, \tau)$. Moreover, the category $\mathcal{O}$ is semisimple.*

We also have

THEOREM 3.18 ([OR]). *If $c$ is generic then the functor $F$ is an equivalence of categories.*

From Theorem 3.17 we can deduce

THEOREM 3.19 ([BEG]). *If $c$ is generic, then $H_c$ is a simple algebra.*

In the case $c = 0$, we get the simplicity of $\mathbf{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathbf{C}[W]$, which is well known.

### 3.8 The Levasseur-Stafford theorem and its generalization

Let us now recall a result of Levasseur and Stafford:

THEOREM 3.20 ([LS]). *If $G$ is a finite group acting on a finite dimensional vector space $V$ over the complex numbers, then the ring $\mathcal{D}(V)^G$ is generated by the subrings $\mathbf{C}[V]^G$ and $\mathbf{C}[V^*]^G$.*

As an example, notice that if we let $\mathbf{Z}/n\mathbf{Z}$ act on the complex line by multiplication by the $n^{\text{th}}$ roots of 1, we deduce that the operator $x\frac{d}{dx}$ can be expressed as a non commutative polynomial in the operators $x^n$ and $\frac{d^n}{dx^n}$, a non-obvious fact. We note also that this theorem has a purely "quantum" nature, i.e. the corresponding "classical" statement, saying that the Poisson algebra $\mathbf{C}[V\times V^*]^G$ is generated, as a Poisson algebra, by $\mathbf{C}[V]^G$ and $\mathbf{C}[V^*]^G$, is in fact false, already for $V=\mathbf{C}$ and $G=\mathbf{Z}/n\mathbf{Z}$.

One can prove a similar result for the algebra $eH_ce$. Namely, recall that the algebra $eH_ce$ contains the subalgebras $\mathbf{C}[\mathfrak{h}]^W$, and $\mathbf{C}[\mathfrak{h}^*]^W$.

THEOREM 3.21 ([BEG]). *If $c$ is generic then the two subalgebras $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ generate $eH_ce$.*

Notice that if $c=0$, then $eH_0e=\mathcal{D}(\mathfrak{h})^W$, so Theorem 3.21 reduces to the Levasseur-Stafford theorem.

REMARK. It is believed that this result holds without the assumption of generic $c$. Moreover, it is known to be true for all $c$ if $W$ is a Weyl group not of type $E$ and $F$, since in this case Wallach proved that the corresponding classical statement for Poisson algebras holds true. Nevertheless, the genericity assumption is needed for the proof, because, similarly to the proof of the Levasseur-Stafford theorem, it is based on the simplicity of $H_c$.

### 3.9 The action of the Cherednik algebra to quasi-invariants

We now go back to the study of $Q_m$. Notice that the algebra $eH_me$ acts on $\mathbf{C}[\mathfrak{h}]^W$, since $e$ gives the $W$-equivariant projection of $\mathbf{C}[\mathfrak{h}]$ onto $\mathbf{C}[\mathfrak{h}]^W$. It is clear that this action is by differential operators. For instance, the subalgebra $\mathbf{C}[\mathfrak{h}]^W\subset eH_me$ acts by multiplication. Also, an element $q\in\mathbf{C}[\mathfrak{h}^*]^W\subset eH_me$ acts via the operator $q(D_{x_1},\dots,D_{x_n})$. By definition this operator coincides with $L_q$ on $\mathbf{C}[\mathfrak{h}]^W$.

The following important theorem shows that this action extends to $Q_m$.

THEOREM 3.22 ([BEG]). *There exists a unique representation of the algebra $eH_me$ on $Q_m$ in which an element $q \in \mathbf{C}[\mathfrak{h}]^W$ acts by multiplication and an element $q \in \mathbf{C}[\mathfrak{h}^*]^W$ by $L_q$.*

*Proof.* Since by Proposition 3.5, $L_q$ preserves $Q_m$, we get a uniquely defined representation of the subalgebra of $eH_me$ generated by $\mathbf{C}[\mathfrak{h}]^W$ and $\mathbf{C}[\mathfrak{h}^*]^W$ on $Q_m$. The result now follows from Theorem 3.21. □

## 3.10 PROOF OF THEOREM 1.8

Finally we can prove Theorem 1.8.

To do this, observe that as an $eH_me$-module, $Q_m$ is in the category $\mathcal{O}(eH_me)$, and $\mathbf{C}[\mathfrak{h}^*]^W$ acts locally nilpotently in $Q_m$ (by degree arguments). We can now apply Theorem 3.18 and Theorem 3.17 and deduce that $Q_m$ is a direct sum of modules of the form $eM(0,\tau)$. As a $\mathbf{C}[\mathfrak{h}] \rtimes \mathbf{C}[W]$-module, $M(0,\tau) = \mathbf{C}[\mathfrak{h}] \otimes \tau$. On the other hand, by Chevalley's theorem, there is an isomorphism $\mathbf{C}[\mathfrak{h}] \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \mathbf{C}[W]$, commuting with the action of $W$ and $\mathbf{C}[\mathfrak{h}]^W$. Thus we get an isomorphisms of $\mathbf{C}[\mathfrak{h}]^W$-modules

$$eM(0,\tau) \simeq (M(0,\tau))^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes (\mathbf{C}[W] \otimes \tau)^W \simeq \mathbf{C}[\mathfrak{h}]^W \otimes \tau\,,$$

proving that $eM(0,\tau)$ and hence $Q_m$ is a free $\mathbf{C}[\mathfrak{h}]^W$-module. □

EXAMPLE 3.23. For $W = \mathbf{Z}/2$ and $\mathfrak{h} = \mathbf{C}$, take the polynomials $1$, $x^{2m+1}$. Notice that $L(1) = L(x^{2m+1}) = 0$ while $s(1) = 1$, $s(x^{2m+1}) = -x^{2m+1}$, $s \in \mathbf{Z}/2$ being the element of order two. It follows that $Q_m$ as a $eH_me$-module is the direct sum of $\mathbf{C}[x^2] \oplus x^{2m+1}\mathbf{C}[x^2]$. These modules are irreducible. Moreover, $\mathbf{C}[x^2] \simeq eM(0,\mathbf{1})$, $x^{2m+1}\mathbf{C}[x^2] \simeq eM(0,\varepsilon)$, $\varepsilon$ being the sign representation.

## 3.11 PROOF OF THEOREM 1.15

Let $I$ be a nonzero two-sided ideal in $\mathcal{D}(X_m)$. First we claim that $I$ nontrivially intersects $Q_m$. Indeed, otherwise let $K \in I$ be a lowest order nonzero element in $I$. Since the order of $K$ is positive, there exists $f \in Q_m$ such that $[K,f] \neq 0$. Then $[K,f] \in I$ is of smaller order than $K$, a contradiction.

Now let $f \in Q_m$ be an element of $I$. Then $g = \prod_{w\in W} {}^w f \in I$. But $g$ is $W$-invariant. This shows that the intersection $J$ of $I$ with the subalgebra $H_m$ in $\mathcal{D}(X_m)$ is nonzero. But $H_m$ is simple by Theorem 3.19, so $J = H_m$. Hence, $1 \in J \subset I$, and $I = \mathcal{D}(X_m)$. □