# 1.1 Définition of quasi-invariants

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proofs are postponed until Lecture 3). In Lecture 2, we explain the origin of the ring of quasi-invariants in the theory of integrable systems, and introduce some tools from integrable systems, such as the Baker-Akhieser function. Finally, in Lecture 3, we develop the theory of the rational Cherednik algebra, the representation-theoretic techniques due to Opdam and Rouquier, and finish the proofs of the geometric statements from Chapter 1.

# 1. Lecture 1

## 1.1 Definition of quasi-invariants

In this lecture we will define the ring of quasi-invariants $Q_m$ and discuss its main properties.

We will work over the field $\mathbf{C}$ of complex numbers. Let $W$ be a finite Coxeter group, i.e. a finite group generated by reflections. Let us denote by $\mathfrak{h}$ its reflection representation. A typical example is the Weyl group of a semisimple Lie algebra acting on a Cartan subalgebra $\mathfrak{h}$. In the case the Lie algebra is $\mathfrak{sl}(n)$, we have that $W$ is the symmetric group $S_n$ on $n$ letters and $\mathfrak{h}$ is the space of diagonal traceless $n \times n$ matrices.

Let $\Sigma \subset W$ denote the set of reflections. Clearly, $W$ acts on $\Sigma$ by conjugation. Let $m\colon \Sigma \to \mathbf{Z}_+$ be a function on $\Sigma$ taking non negative integer values, which is $W$-invariant. The number of orbits of $W$ on $\Sigma$ is generally very small. For example, if $W$ is the Weyl group of a simple Lie algebra of ADE type, then $W$ acts transitively on $\Sigma$, so $m$ is a constant function.

For each reflection $s \in \Sigma$, choose $\alpha_s \in \mathfrak{h}^* - \{0\}$ so that, for $x \in \mathfrak{h}$, $\alpha_s(sx) = -\alpha_s(x)$ (this means that the hyperplane given by the equation $\alpha_s = 0$ is the reflection hyperplane for $s$).

Definition 1.1 ([CV1, CV2]). A polynomial $q \in \mathbf{C}[\mathfrak{h}]$ is said to be *$m$-quasi-invariant* with respect to $W$ if, for any $s \in \Sigma$, the polynomial $q(x) - q(sx)$ is divisible by $\alpha_s(x)^{2m_s+1}$.

We will denote by $Q_m$ the space of $m$-quasi-invariant polynomials with respect to $W$.

Notice that every element of $\mathbf{C}[\mathfrak{h}]$ is a $0$-quasi-invariant, and that every $W$-invariant is an $m$-quasi-invariant for any $m$. Indeed if $q \in \mathbf{C}[\mathfrak{h}]^W$, then we have $q(x) - q(sx) = 0$ for all $s \in \Sigma$, and $0$ is divisible by all powers of $\alpha_s(x)$. Thus in a way, $\mathbf{C}[\mathfrak{h}]^W$ can be viewed as the set of $\infty$-quasi-invariants.

EXAMPLE 1.2. The group $W = \mathbf{Z}/2$ acts on $\mathfrak{h} = \mathbf{C}$ by $s(v) = -v$. In this case $m$ is a non negative integer and $\Sigma = \{s\}$. So this definition says that $q$ is in $Q_m$ iff $q(x) - q(-x)$ is divisible by $x^{2m+1}$. It is very easy to write a basis of $Q_m$. It is given by the polynomials $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$.

## 1.2 ELEMENTARY PROPERTIES OF $Q_m$

Some elementary properties of $Q_m$ are collected in the following proposition.

PROPOSITION 1.3 (see [FV] and references therein).

1) $\mathbf{C}[\mathfrak{h}]^W \subset Q_m \subseteq \mathbf{C}[\mathfrak{h}]$, $Q_0 = \mathbf{C}[\mathfrak{h}]$, $Q_m \subset Q_{m'}$ *if* $m \geq m'$, $\bigcap_m Q_m = \mathbf{C}[\mathfrak{h}]^W$.
2) $Q_m$ *is a graded subalgebra of* $\mathbf{C}[\mathfrak{h}]$.
3) *The fraction field of* $Q_m$ *is equal to* $\mathbf{C}(\mathfrak{h})$.
4) $Q_m$ *is a finite* $\mathbf{C}[\mathfrak{h}]^W$*-module and a finitely generated algebra.* $\mathbf{C}[\mathfrak{h}]$ *is a finite* $Q_m$*-module.*

*Proof.* 1) is immediate and has already been mentioned in 1.1.

2) Clearly $Q_m$ is closed under addition. Let $p, q \in Q_m$. Let $s \in \Sigma$. Then

$$p(x)q(x) - p(sx)q(sx) = (p(x) - p(sx))q(x) + p(sx)(q(x) - q(sx)).$$

Since both $p(x) - p(sx)$ and $q(x) - q(sx)$ are divisible by $\alpha_s^{2m_s+1}$, we deduce that $p(x)q(x) - p(sx)q(sx)$ is also divisible by $\alpha_s^{2m_s+1}$, proving the claim.

3) Consider the polynomial

$$\delta_{2m+1}(x) = \prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}.$$

This polynomial is uniquely defined up to scaling. One has $\delta_{2m+1}(sx) = -\delta_{2m+1}(x)$ for each $s \in \Sigma$, hence $\delta_{2m+1} \in Q_m$. Take $f(x) \in \mathbf{C}[\mathfrak{h}]$. We claim that $f(x)\delta_{2m+1}(x) \in Q_m$. As a matter of fact,

$$f(x)\delta_{2m+1}(x) - f(sx)\delta_{2m+1}(sx) = (f(x) + f(sx))\delta_{2m+1}(x),$$

and by its definition $\delta_{2m+1}(x)$ is divisible by $\alpha_s(x)^{2m_s+1}$ for all $s \in \Sigma$. This implies 3).

4) By Hilbert's theorem on the finiteness of invariants, we get that $\mathbf{C}[\mathfrak{h}]^W$ is a finitely generated algebra over $\mathbf{C}$ and $\mathbf{C}[\mathfrak{h}]$ is a finite $\mathbf{C}[\mathfrak{h}]^W$-module and hence a finite $Q_m$-module, proving the second part of 4).

Now $Q_m \subset \mathbf{C}[\mathfrak{h}]$ is a submodule of the finite module $\mathbf{C}[\mathfrak{h}]$ over the Noetherian ring $\mathbf{C}[\mathfrak{h}]^W$. Hence it is finite. This immediately implies that $Q_m$ is a finitely generated algebra over $\mathbf{C}$. $\square$