

# 5. The basic gerbe over a compact simple Lie group

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## 5. THE BASIC GERBE OVER A COMPACT SIMPLE LIE GROUP

In this section we explain our construction of the basic gerbe over a compact, simple, simply connected Lie group.

## 5.1 NOTATION

Let  $G$  be a compact, simple, simply connected Lie group, with Lie algebra  $\mathfrak{g}$ . For any action of  $G \times M \rightarrow M$ ,  $(g, m) \mapsto g.m$  on a manifold  $M$ , we will denote by  $G_m$  the stabilizer group of a point  $m \in M$ . If  $M = G$  or  $M = \mathfrak{g}$ , we will always consider the adjoint action of  $G$  unless specified otherwise. For instance,  $G_g$  for denotes the centralizer of an element  $g \in G$ .

Choose a maximal torus  $T$  of  $G$ , with Lie algebra  $\mathfrak{t}$ . Let  $\Lambda = \ker(\exp|_{\mathfrak{t}})$  be the integral lattice and  $\Lambda^* \subset \mathfrak{t}^*$  its dual, the (real) weight lattice. Equivalently,  $\Lambda$  is characterized as the lattice generated by the coroots  $\check{\alpha}$  for the (real) roots  $\alpha$ . Recall that the *basic inner product*  $\cdot$  on  $\mathfrak{g}$  is the unique invariant inner product such that  $\check{\alpha} \cdot \check{\alpha} = 2$  for all long roots  $\alpha$ . Throughout this paper, we will use the basic inner product to identify  $\mathfrak{g}^* \cong \mathfrak{g}$ . Choose a collection of simple roots  $\alpha_1, \dots, \alpha_d \in \Lambda^*$  and let  $\mathfrak{t}_+ = \{\xi \mid \alpha_j \cdot \xi \geq 0, j = 1, \dots, d\}$  be the corresponding positive Weyl chamber. The fundamental alcove  $\mathfrak{A}$  is the subset cut out from  $\mathfrak{t}_+$  by the additional inequality  $\alpha_0 \cdot \xi \geq -1$  where  $\alpha_0$  is the lowest root.

The fundamental alcove parametrizes conjugacy classes in  $G$ , in the sense that each conjugacy class contains a unique point  $\exp \xi$  with  $\xi \in \mathfrak{A}$ . The quotient map will be denoted  $q: G \rightarrow \mathfrak{A}$ . Let  $\mu_0, \dots, \mu_d$  be the vertices of  $\mathfrak{A}$ , with  $\mu_0 = 0$ . For any  $I \subseteq \{0, \dots, d\}$ , all group elements  $\exp \xi$  with  $\xi$  in the open face spanned by  $\mu_j$  with  $j \in I$  have the same centralizer, denoted  $G_I$ . In particular,  $G_j$  will denote the centralizer of  $\exp \mu_j$ .

For each  $j$  let  $\mathfrak{A}_j \subset \mathfrak{A}$  be the open star at  $\mu_j$ , i.e. the union of all open faces containing  $\mu_j$  in their closure. Put differently,  $\mathfrak{A}_j$  is the complement of the closed face opposite to the vertex  $\mu_j$ . We will work with the open cover of  $G$  given by the pre-images,  $V_j = q^{-1}(\mathfrak{A}_j)$ . More generally let  $\mathfrak{A}_I = \bigcap_{j \in I} \mathfrak{A}_j$ , and  $V_I := q^{-1}(\mathfrak{A}_I)$ . The flow-out  $S_I = G_I \cdot \exp(\mathfrak{A}_I)$  of  $\exp(\mathfrak{A}_I) \subset T$  under the action of  $G_I$  is an open subset of  $G_I$ , and is a slice for the conjugation action of  $G$ . That is,

$$G \times_{G_I} S_I = V_I.$$

We let  $\pi_I: V_I \rightarrow G/G_I$  denote the projection to the base.

## 5.2 THE BASIC 3-FORM ON $G$

Let  $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$  be the left- and right-invariant Maurer-Cartan forms on  $G$ , respectively. The 3-form  $\eta \in \Omega^3(G)$  given by<sup>3)</sup>

$$\eta = \frac{1}{12} \theta^L \cdot [\theta^L, \theta^L] = \frac{1}{12} \theta^R \cdot [\theta^R, \theta^R]$$

is closed, and has a closed equivariant extension  $\eta_G \in \Omega_G^3(G)$  given by

$$\eta_G(\xi) := \eta - \frac{1}{2}(\theta^L + \theta^R) \cdot \xi.$$

Their cohomology classes represent generators of  $H^3(G, \mathbf{Z}) = \mathbf{Z}$  and  $H_G^3(G, \mathbf{Z}) = \mathbf{Z}$ , respectively. The pull-back of  $\eta_G$  to any conjugacy class  $\iota_C: C \hookrightarrow G$  is exact. In fact, let  $\omega_C \in \Omega^2(C)^G \subset \Omega_G^2(C)$  be the invariant 2-form given on generating vector fields  $\xi_C, \xi'_C$  for  $\xi, \xi' \in \mathfrak{g}$  by the formula

$$\omega_C(\xi_C(g), \xi'_C(g)) = \frac{1}{2} \xi \cdot (\text{Ad}_g - \text{Ad}_{g^{-1}}) \xi'.$$

Then [1, 16]

$$d_G \omega_C + \iota_C^* \eta_G = 0.$$

We will now show that  $\eta_G$  is exact over each of the open subsets  $V_j$ . Let  $C_j = q^{-1}(\mu_j) \subset V_j$  be the conjugacy classes corresponding to the vertices.

LEMMA 5.1. *The linear retraction*

$$[0, 1] \times \mathfrak{A}_j \rightarrow \mathfrak{A}_j, \quad (t, \mu_j + \zeta) \mapsto \mu_j + (1 - t) \zeta$$

of  $\mathfrak{A}_j$  onto the vertex  $\mu_j$  lifts uniquely to a smooth  $G$ -equivariant retraction from  $V_j$  onto  $C_j$ .

*Proof.* Recall that the slice  $S_j$  is an open neighborhood of  $\exp(\mu_j)$  in  $G_j$ . Any  $G_j$ -equivariant retraction from  $S_j$  onto  $\exp \mu_j$  extends uniquely to a  $G$ -equivariant retraction from  $V_j = G \times_{G_j} S_j$  onto  $C_j$ . Note that  $S'_j = G_j \cdot (\mathfrak{A}_j - \mu_j)$  is a star-shaped open neighborhood of 0 in  $\mathfrak{g}_j$ , and that  $S'_j \rightarrow S_j, \zeta \mapsto \exp(\mu_j) \exp(\zeta)$  is a  $G_j$ -equivariant diffeomorphism. The linear retraction of  $S'_j$  onto the origin gives the desired retraction of  $S_j$ . Uniqueness is clear, since the retraction has to preserve  $\exp(\mathfrak{A}_j) \subset V_j$ , by equivariance.

<sup>3)</sup> For  $\mathfrak{g}$ -valued forms  $\beta_1, \beta_2$ , the bracket  $[\beta_1, \beta_2]$  denotes the  $\mathfrak{g}$ -valued form obtained by applying the Lie bracket  $[\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  to the  $\mathfrak{g} \otimes \mathfrak{g}$ -valued form  $\beta_1 \wedge \beta_2$ .

Let

$$\mathbf{h}_j: \Omega^p(V_j) \rightarrow \Omega^p([0, 1] \times V_j) \rightarrow \Omega^{p-1}(V_j)$$

be the de Rham homotopy operator for this retraction, given (up to a sign) by pull-back under the retraction, followed by integration over the fibers of  $[0, 1] \times V_j \rightarrow V_j$ . It has the property

$$(5.1) \quad d_G \mathbf{h}_j + \mathbf{h}_j d_G = \text{Id} - \pi_j^* \iota_j^*$$

where  $\iota_j: C_j \rightarrow V_j$  is the inclusion and  $\pi_j: V_j = G \times_{G_j} S_j \rightarrow G/G_j = C_j$  the projection. Let  $(\varpi_j)_G = \mathbf{h}_j \eta_G - \pi_j^* \omega_{C_j} \in \Omega_G^2(V_j)$ , and write  $(\varpi_j)_G = \varpi_j - \Psi_j$  where  $\varpi_j \in \Omega^2(V_j)$  and  $\Psi_j \in \Omega^0(V_j, \mathfrak{g})$ .

**PROPOSITION 5.2.** *The equivariant 2-form  $(\varpi_j)_G = \varpi_j - \Psi_j$  has the following properties.*

(a)  $d_G(\varpi_j)_G = \eta_G$ .

(b) *The pull-back of  $(\varpi_j)_G$  to a conjugacy class  $C \subset V_j$  is given by*

$$\iota_C^*(\varpi_j)_G = \Psi_j^*(\omega_{\mathcal{O}})_G - \omega_C,$$

where  $(\omega_{\mathcal{O}})_G$  is the equivariant symplectic form on the adjoint orbit  $\mathcal{O} = \Psi_j(C)$ ,

(c) *The pull-back of  $\Psi_j$  to the conjugacy class  $C_j$  vanishes. In fact,  $\Psi_j(\exp \xi) = \xi - \mu_j$  for all  $\xi \in \mathfrak{A}_j$ .*

(d) *Over each intersection  $V_{ij} = V_i \cap V_j$ , the difference  $\Psi_i - \Psi_j$  takes values in the adjoint orbit  $\mathcal{O}_{ij}$  through  $\mu_j - \mu_i \in \mathfrak{g} \cong \mathfrak{g}^*$ . Furthermore,*

$$(\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G$$

where  $p_{ij}: V_{ij} \rightarrow \mathcal{O}_{ij}$  is the map defined by  $\Psi_i - \Psi_j$ , and  $(\omega_{\mathcal{O}_{ij}})_G$  is the equivariant symplectic form on the orbit.

*Proof.* (a) holds by construction. (b) follows from the observation that  $\iota_C^*(\varpi_j)_G + \omega_C$  is an equivariantly closed 2-form on  $C_j$ , with  $\Psi_j$  as its moment map. To prove (c) we note that since the retraction is equivariant, we have  $\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* = (\exp|_{\mathfrak{A}_j})^* \circ \mathbf{h}_j$  where  $(\exp|_{\mathfrak{A}_j})^*$  is pull-back to  $\mathfrak{A}_j \subset \mathfrak{t}$  and where  $\tilde{\mathbf{h}}_j$  is the homotopy operator for the linear retraction of  $\mathfrak{t}$  onto  $\{\mu_j\}$ . Let  $\nu: \mathfrak{A}_j \rightarrow \mathfrak{t}$  be the coordinate function (inclusion). Then

$$\tilde{\mathbf{h}}_j \circ (\exp|_{\mathfrak{A}_j})^* \frac{1}{2}(\theta^L + \theta^R) = \tilde{\mathbf{h}}_j \circ d\nu = \nu - \mu_j,$$

proving that  $(\exp|_{\mathfrak{A}_j})^* \Psi_j = \nu - \mu_j$ . This yields (c), by equivariance. For  $\nu \in \mathfrak{A}_{ij}$  we have, using (c),

$$(\Psi_i - \Psi_j)(\exp \nu) = (\nu - \mu_i) - (\nu - \mu_j) = \mu_j - \mu_i.$$

By equivariance, it follows that  $\Psi_i - \Psi_j$  takes values in the adjoint orbit through  $\mu_j - \mu_i$ . The difference  $\varpi_i - \varpi_j$  vanishes on the maximal torus  $T$ , and is therefore determined by its contractions with generating vector fields. Since  $\Psi_i - \Psi_j$  is a moment map for  $\varpi_i - \varpi_j$ , it follows that  $\varpi_i - \varpi_j$  equals the pull-back of the symplectic form on  $G \cdot (\mu_j - \mu_i)$ .

### 5.3 THE SPECIAL UNITARY GROUP

For the special unitary group  $G = \mathrm{SU}(d+1)$ , the construction of the basic gerbe simplifies due to the fact that in this case all vertices  $\mu_j$  of the alcove are contained in the weight lattice. In fact the gerbe is presented as a Chatterjee-Hitchin gerbe for the cover  $\mathcal{V} = \{V_i, i = 0, \dots, d\}$ .

For each weight  $\mu \in \Lambda^* \subset \mathfrak{t} \subset \mathfrak{g}$ , let  $G_\mu$  be its stabilizer for the adjoint action and let  $\mathbf{C}_\mu$  the 1-dimensional  $G_\mu$ -representation with infinitesimal character  $\mu$ . Let the line bundle  $L_\mu = G \times_{G_\mu} \mathbf{C}_\mu$  equipped with the unique left-invariant connection  $\nabla$ . Then  $L_\mu$  is a  $G$ -equivariant pre-quantum line bundle for the orbit  $\mathcal{O} = G \cdot \mu$ . That is,

$$\frac{i}{2\pi} \mathrm{curv}_G(\nabla) = (\omega_{\mathcal{O}})_G := \omega_{\mathcal{O}} - \Phi_{\mathcal{O}}$$

where  $\omega_{\mathcal{O}}$  is the symplectic form and  $\Phi_{\mathcal{O}}: \mathcal{O} \hookrightarrow \mathfrak{g}^*$  is the moment map given as inclusion.

In particular, in the case of  $\mathrm{SU}(d+1)$  all orbits  $\mathcal{O}_{ij} = G \cdot (\mu_j - \mu_i)$  carry  $G$ -equivariant pre-quantum line bundles. Recall the fibrations  $p_{ij}: V_{ij} \rightarrow \mathcal{O}_{ij}$  defined by  $\Psi_i - \Psi_j$ , and let

$$L_{ij} = p_{ij}^*(L_{\mu_j - \mu_i}),$$

equipped with the pull-back connection. For any triple intersection  $V_{ijk} = G \times_{G_{ijk}} S_{ijk}$ , the tensor product  $(\delta L)_{ijk} = L_{jk} L_{ik}^{-1} L_{ij}$  is the pull-back of the line bundle over  $G/G_{ijk}$ , defined by the zero weight

$$(\mu_k - \mu_j) - (\mu_k - \mu_i) + (\mu_j - \mu_i) = 0$$

of  $G_{ijk}$ . It is hence canonically trivial, with  $(\delta \nabla)_{ijk}$  the trivial connection. The trivializing section  $t_{ijk} = 1$  satisfies  $\delta t = 1$  and  $(\delta \nabla)t = 0$ . Take  $(B_j)_G = (\varpi_j)_G$ . Then

$$(B_j)_G - (B_i)_G = (\varpi_j)_G - (\varpi_i)_G = -p_{ij}^*(\omega_{\mathcal{O}_{ij}})_G = \frac{1}{2\pi i} \mathrm{curv}_G(\nabla^{L_{ij}}).$$

Thus  $\mathcal{G} = (\mathcal{V}, L, t)$  is a equivariant gerbe with connection  $(\nabla, B)$ . Since

$$d_G(B_j)_G = d_G(\varpi_j)_G = \eta_G|_{V_j},$$

this is the basic gerbe for  $SU(d+1)$ . The transition line bundles  $L_{ij}$  may be expressed in terms of eigenspace line bundles, leading to the description of the basic gerbe from the introduction.

REMARK 5.3. This description of the basic gerbe over the special unitary group was found independently by Gawędzki-Reis [13], who also discuss the much more difficult case of quotients of  $SU(d+1)$  by subgroups of the center.

A similar construction works for the group  $C_d = Sp(d)$ , the only case besides  $A_d = SU(d+1)$  for which the vertices of the alcove are in the weight lattice. The following table lists, for all simply connected compact simple groups, the smallest integer  $k_0 > 0$  such that  $k_0\mathfrak{A}$  is a weight lattice polytope<sup>4</sup>). The construction for  $SU(d+1)$  generalizes to describe the  $k_0$ -th power of the basic gerbe in all cases.

(5.2)

$G$	$A_d$	$B_d$	$C_d$	$D_d$	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
$k_0$	1	2	1	2	6	12	60	6	2

#### 5.4 THE BASIC GERBE FOR GENERAL SIMPLE, SIMPLY CONNECTED $G$

The extra difficulty for the groups with  $k_0 > 1$  comes from the fact that the pull-back maps  $H_G^3(G, \mathbf{Z}) \rightarrow H_G^3(C_j, \mathbf{Z}) \cong H_G^3(V_j, \mathbf{Z})$  may be a non-zero torsion class, in general. In this case the restriction of the basic gerbe to  $V_j$  will be non-trivial. Our strategy for the general case is to first construct equivariant bundle gerbes over  $V_j$ , and then glue the local data as explained in Section 4.

The centralizers  $G_g$  of elements  $g \in G$  are always connected [11, Corollary (3.15)] but need not be simply-connected. The conjugacy classes  $C_j = q^{-1}(\mu_j)$  corresponding to the vertices of the alcove are exactly the conjugacy classes of elements for which the centralizer is semi-simple. Since

$$H_G^3(C_j, \mathbf{Z}) = H_G^3(G/G_j, \mathbf{Z}) = H_{G_j}^3(\text{pt}, \mathbf{Z}),$$

we see that the torsion problem described above is related to a possibly non-trivial central extension of the centralizers  $G_j$  of  $\exp(\mu_j)$  by the circle  $U(1)$ .

<sup>4</sup>) This information is extracted from the tables in Bourbaki [5]. Letting  $w_1, \dots, w_d$  be the fundamental weights, one determines  $k_0$  as the least common multiple of the numbers  $\alpha_{\max} \cdot w_j$ , using the basic inner product defined by  $\alpha_{\max} \cdot \alpha_{\max} = 2$ . The number  $k_0$  is equal to the smallest Dynkin index of a representation  $G \rightarrow SU(n)$ , see [28, p. 128] where the same table appears in a different context.

PROPOSITION 5.4. *Any vertex  $\mu_j$  of the alcove  $\mathfrak{A}$  is in the dual of the co-root lattice for the corresponding centralizer  $G_j$ . It hence defines a homomorphism  $\varrho_j \in \text{Hom}(\pi_1(G_j), \text{U}(1))$ , or equivalently a central extension of  $G_j$  by  $\text{U}(1)$ .*

*Proof.* Let  $\tilde{G}_j$  be the universal cover of  $G_j$ . A system of simple roots for  $\tilde{G}_j$  is given by the list of all  $\alpha_i$  ( $i = 0, \dots, d$ ) with  $j \neq i$ . The lattice  $\Lambda_j$  is spanned by the corresponding coroots  $\check{\alpha}_i$ . To show that  $\mu_j$  is in the dual of the co-root lattice, we have to verify that  $\langle \mu_j, \check{\alpha}_i \rangle \in \mathbb{Z}$  for  $i \neq j$ . For  $i \neq 0, j$  this is obvious since  $\mu_j(\check{\alpha}_i) = 0$ . For  $i = 0$ , we have  $\|\check{\alpha}_0\|^2 = 2$ , and therefore  $\check{\alpha}_0 = \alpha_0$  and  $\mu_j(\check{\alpha}_0) = \alpha_0(\mu_j) = -1$ .

Recall that for  $i \neq j$ ,  $G_{ij}$  is the centralizer of points  $\exp \mu$  with  $\mu = t\mu_j + (1-t)\mu_i$  for some  $0 < t < 1$ . Let  $\varrho_{ij} \in \text{Hom}(\pi_1(G_{ij}), \text{U}(1))$  be the quotient of  $\pi_1(G_{ij}) \rightarrow \pi_1(G_j) \xrightarrow{\varrho_j} \text{U}(1)$  by the homomorphism  $\pi_1(G_{ij}) \rightarrow \pi_1(G_i) \xrightarrow{\varrho_i} \text{U}(1)$ .

LEMMA 5.5. *The difference  $\mu_j - \mu_i \in \mathfrak{g}_{ij}$  is fixed under  $G_{ij}$ , and  $\varrho_{ij} \in \text{Hom}(\pi_1(G_{ij}), \text{U}(1))$  is its image under the exact sequence (3.2) for  $K = G_{ij}$ .*

*Proof.* Since  $G_{ij}$  fixes the curve  $g(t) = \exp(t\mu_j + (1-t)\mu_i) = \exp(\mu_i)\exp(t(\mu_j - \mu_i))$ , it stabilizes the Lie algebra element  $\mu_j - \mu_i$ . The second claim is immediate from the definition.

We are now in position to explain our construction of the basic gerbe in the general case. For all  $I \subset \{0, \dots, d\}$  let  $X_I \rightarrow V_I$  be the  $G$ -equivariant principal  $G_I$ -bundle,

$$X_I = G \times S_I \rightarrow V_I = G \times_{G_I} S_I.$$

$X_I$  is the pull-back of the  $G_I$ -bundle  $G \rightarrow G/G_I$ , and in particular carries a  $G$ -invariant connection  $\theta_I$  obtained by pull-back of the unique  $G$ -invariant connection on that bundle. For  $I \supset J$  there are natural  $G$ -equivariant inclusions  $f_I^J: X_I \rightarrow X_J$ , and these are compatible as in Section 4. The homomorphisms  $\varrho_j: \pi_1(G_j) \rightarrow \text{U}(1)$  define flat,  $G$ -equivariant bundle gerbes  $\mathcal{G}_j = (X_j, L_j, t_j)$  over  $V_j$ .

The quotient of the two gerbes on  $V_{ij}$ , obtained by pulling back  $\mathcal{G}_i, \mathcal{G}_j$  to  $X_{ij}$ , is just the gerbe defined by the homomorphism  $\varrho_{ij}: \pi_1(G_{ij}) \rightarrow \text{U}(1)$ . By Lemma 5.5 and Proposition 3.2(b), it follows that this quotient gerbe has

a distinguished, equivariant pseudo-line bundle  $(E_{ij}, s_{ij})$  (where  $E_{ij}$  is trivial), with connection  $\nabla^{E_{ij}}$  induced from the connection  $\theta_{ij}$ . From the definition of  $\theta_{ij}$ , it follows that the equivariant error 2-form for this connection is the pull-back of the equivariant symplectic form on the coadjoint orbit through  $\mu_j - \mu_i$ .

We now modify the bundle gerbe connection by adding the equivariant 2-form  $(\varpi_j)_G \in \Omega_G^2(V_j)$  to the gerbe connection. Proposition 5.2(d) shows that the equivariant error 2-form of  $\nabla^{E_{ij}}$  with respect to the new gerbe connection vanishes. The other conditions from the gluing construction in §4 are trivially satisfied. Since the equivariant 3-curvature for the new gerbe connection on  $\mathcal{G}_j$  is  $d_G(\varpi_j)_G = \eta_G|_{V_j}$ , we have constructed an equivariant bundle gerbe with connection, with equivariant curvature-form  $\eta_G$ .

REMARK 5.6. For  $G = \mathrm{SU}(d+1)$  this construction reduces to the construction in terms of transition line bundles: All  $L_i, t_i, E_{ij}, u_{ijk}$  are trivial in this case, hence the entire information on the gerbe resides in the functions  $s_{ij}: (X_{ij})^{[2]} \rightarrow \mathrm{U}(1)$  defined by the differences  $\mu_j - \mu_i$ . The condition  $\delta s_{ij} = 1$  for these functions means that  $s_{ij}$  defines a line bundle  $L_{ij}$  over  $V_{ij}$ , as remarked at the beginning of Section 2.2. The condition  $s_{ij}s_{jk}s_{ki} = 1$  over  $X_{ijk}$  is the compatibility condition over triple intersections.

## 6. PRE-QUANTIZATION OF CONJUGACY CLASSES

It is a well-known fact from symplectic geometry that a coadjoint orbit  $\mathcal{O} = G \cdot \mu$  through  $\mu \in \mathfrak{t}_+^*$  has integral symplectic form, i.e. admits a pre-quantum line bundle, if and only if  $\mu$  is in the weight lattice  $\Lambda^*$ . The analogous question for conjugacy classes reads: For which  $\mu \in \mathfrak{A}$  and  $m \in \mathbb{N}$  does the pull-back of the  $m$ th power of the basic gerbe  $\mathcal{G}^m$  to the conjugacy class  $\mathcal{C} = G \cdot \exp(\mu)$  admit a pseudo-line bundle, with  $m\omega_{\mathcal{C}}$  as its error 2-form? For any positive integer  $m > 0$  let

$$\Lambda_m^* = \Lambda^* \cap m\mathfrak{A}$$

be the set of level  $m$  weights. As is well-known [26], the set  $\Lambda_m^*$  parametrizes the positive energy representations of the loop group  $LG$  at level  $m$ .

THEOREM 6.1. *The restriction of  $\mathcal{G}^m$  to a conjugacy class  $\mathcal{C}$  admits a pseudo-line bundle  $\mathcal{L}$  with connection, with error 2-form  $m\omega_{\mathcal{C}}$ , if and only if  $\mathcal{C} = G \cdot \exp(\mu/m)$  with  $\mu \in \Lambda_m^*$ . Moreover  $\mathcal{L}$  has an equivariant extension in this case, with  $m\omega_{\mathcal{C}}$  as its equivariant error 2-form.*