3. Gerbes from principal bundles

REMARK 2.8. As pointed out in Mathai-Stevenson [21], this notion of equivariant bundle gerbe is sometimes 'really too strong': For instance, if $X = \coprod U_a$, for an open cover $\mathcal{U} = \{U_a, a \in A\}$, a G-action on X would amount to the cover being G-invariant. Brylinski [9] on the other hand gives a definition of equivariant Chatterjee-Hitchin gerbes that does not require invariance of the cover.

To define equivariant connections and curvature, we will need some notions from equivariant de Rham theory [15]. Recall that for a compact group G, the equivariant cohomology $H_G^\bullet(M, \mathbf{R})$ may be computed from Cartan's complex of equivariant differential forms $\Omega_G^\bullet(M)$, consisting of G-equivariant polynomial maps $\alpha\colon \mathfrak{g} \to \Omega(M)$. The grading is the sum of the differential form degree and twice the polynomial degree, and the differential reads

$$(\mathrm{d}_G\, \alpha)(\xi) = \mathrm{d}\, \alpha(\xi) - \iota(\xi_M)\alpha(\xi)\,,$$

where $\xi_M = \frac{d}{dt}|_{t=0} \exp(-t\xi)$ is the generating vector field corresponding to $\xi \in \mathfrak{g}$. Given a G-equivariant connection ∇^L on an equivariant line bundle, one defines [3, Chapter 7] a d_G-closed equivariant curvature $\mathrm{curv}_G(\nabla^L) \in \Omega_G^2(M)$.

A equivariant connection on a G-equivariant bundle gerbe (X, L, t) over M is a pair (∇^L, B_G), where ∇^L is an invariant connection and $B_G \in \Omega_G^2(X)$ an equivariant 2-form, such that $\delta\nabla^L t = 0$ and $\delta B_G = \frac{1}{2\pi i}\,\mathrm{curv}_G(\nabla^L)$. Its equivariant 3-curvature $\eta_G \in \Omega_G^3(M)$ is defined by $\pi^*\eta_G = \mathrm{d}_G B_G$. Given an *invariant* pseudo-line bundle connection ∇^E on a equivariant pseudo-line bundle (E, s), one defines the equivariant error 2-form ω_G by

$$\pi^*\omega_G = \frac{1}{2\pi i}\,\mathrm{curv}_G(\nabla^E) - B_G\,.$$

Clearly, $\mathrm{d}_G\,\omega_G + \eta_G = 0$.

3. GERBES FROM PRINCIPAL BUNDLES

The following well-known example [7], [24] of a gerbe will be important for our construction of the basic gerbe over G. Suppose $\mathrm{U}(1) \to \widehat{K} \to K$ is a central extension, and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. Given a principal K-bundle $\pi\colon P \to B$, one constructs a bundle gerbe (P, L, t), sometimes called the lifting bundle gerbe. Observe that

$$E_n P = P \times_K E_n K\,,$$

which we may view as a fiber bundle over B but also as a fiber bundle $E_nK \times_K P$ over B_nK. Let

$$f_\bullet \colon E_\bullet P \to B_\bullet K \tag{3.1}$$

be the bundle projection. Then $L = f_1^*\Gamma$, $t = f_2^*\tau$ defines a bundle gerbe (P, L, t). A pseudo-line bundle for this bundle gerbe is equivalent to a lift of the structure group to $\widehat{K}$: Indeed if $\widehat{P}$ is a principal $\widehat{K}$-bundle lifting P, consider the associated bundle $E = \widehat{P} \times_{\mathrm{U}(1)} \mathbf{C}$. From the action map $\widehat{K} \times \widehat{P} \to \widehat{P}$ one obtains an isomorphism $\Gamma_k \otimes E_p \cong E_{k.p}$, or equivalently a section s of $\delta E^{-1} \otimes L$. One checks that $\delta s = t$, so that (E, s) is a pseudo-line bundle. Conversely, the bundle $\widehat{P}$ is recovered as the unit circle bundle in E, and s defines an action of $\widehat{K}$ lifting the action of K. See Gomi [14] for a detailed construction of bundle gerbe connections on (P, L, t).

REMARK 3.1. To obtain a Chatterjee-Hitchin gerbe from this bundle gerbe, we must choose a cover $\mathcal{U}$ of M such that P is trivial over each $U_a \in \mathcal{U}$. Any choice of trivialization gives a simplicial map $\mathcal{U}_\bullet M \to E_\bullet P$, and we pull back the bundle gerbe under this map. More directly, the local trivializations give rise to a 'classifying map' $\chi_\bullet \colon \mathcal{U}_\bullet M \to B_\bullet K$ (see [23]), and the Chatterjee-Hitchin gerbe is defined as the pull-back of (Γ, τ) under this map.

Suppose the group K is compact and connected. After pulling back to the universal cover $\widetilde{K}$, every central extension $\mathrm{U}(1) \to \widehat{K} \to K$ becomes trivial. It follows that every central extension of K by $\mathrm{U}(1)$ is of the form

$$\widehat{K} = \widetilde{K} \times_{\pi_1(K)} \mathrm{U}(1),$$

where $\pi_1(K) \subset \widetilde{K}$ acts on $\mathrm{U}(1)$ via some homomorphism $\varrho \in \mathrm{Hom}(\pi_1(K), \mathrm{U}(1))$. The choice of ϱ for a given extension is equivalent to the choice of a flat $\widehat{K}$-invariant connection on the principal $\mathrm{U}(1)$-bundle $\widehat{K} \to K$. The central extension is isomorphic to the *trivial* extension if and only if ϱ extends to a homomorphism $\widetilde{\varrho} \colon \widetilde{K} \to \mathrm{U}(1)$, and the choice of any such $\widetilde{\varrho}$ is equivalent to a choice of trivialization. Using the natural map from $(\mathfrak{k}^*)^K = \mathrm{Hom}(\widetilde{K}, \mathbf{R})$ onto $\mathrm{Hom}(\widetilde{K}, \mathrm{U}(1))$ this gives an exact sequence of Abelian groups

$$(\mathfrak{k}^*)^K \to \mathrm{Hom}(\pi_1(K), \mathrm{U}(1)) \to \{\text{central extensions of } K \text{ by } \mathrm{U}(1)\} \to 1. \tag{3.2}$$

Suppose K is semi-simple (so that $(\mathfrak{k}^*)^K = 0$), and T is a maximal torus in K. Let $\widetilde{T} \subset \widetilde{K}$ be the maximal torus given as the pre-image of T. Let $\Lambda_K, \widetilde{\Lambda}_K \subset \mathfrak{t}$ be the integral lattices of $T, \widetilde{T}$. The lattice $\widetilde{\Lambda}_K$ is equal to the co-root lattice of K, and $\pi_1(K) = \Lambda_K/\widetilde{\Lambda}_K$ (cf. [6, Theorem V.7.1]). Therefore, if K is semi-simple,

$$\{\text{central extensions of } K \text{ by } \mathrm{U}(1)\} = \mathrm{Hom}(\pi_1(K), \mathrm{U}(1)) = \widetilde{\Lambda}_K^*/\Lambda_K^*,$$

the quotient of the dual of the co-root lattice by the weight lattice.

PROPOSITION 3.2. *Suppose K is a compact, connected Lie group and $\pi: P \to M$ a principal K-bundle.*

(a) *Any $\varrho \in \mathrm{Hom}(\pi_1(K), \mathrm{U}(1))$ defines a bundle gerbe (P, L, t) over M, together with a gerbe connection (∇^L, B) where $B = 0$. In particular this gerbe is* flat.

(b) *If ϱ is the image of $\mu \in (\mathfrak{k}^*)^K$, there is a distinguished pseudo-line bundle $\mathcal{L} = (E, s)$ for this gerbe, with E a trivial line bundle. Any principal connection $\theta \in \Omega^1(P, \mathfrak{k})$ defines a connection on $\mathcal{L}$, with error 2-form $\omega \in \Omega^2(M)$ given by $\pi^*\omega = \langle \mu, F^\theta \rangle \in \Omega^2(M)$, where F^θ is the curvature.*

Proof. Let $\mathrm{U}(1) \to \widehat{K} \to K$ be the central extension defined by ϱ, and (Γ, τ) the corresponding simplicial gerbe over $B_\bullet K$. As remarked above, ϱ defines a flat connection on $\widehat{K} \to K$, hence also a flat connection ∇^Γ on the line bundle $\Gamma \to B_1 K$. Then $(\nabla^\Gamma, 0)$ is a connection on the simplicial gerbe (Γ, τ). Pulling back under the map $f_\bullet$ (cf. (3.1)) we obtain a connection $(\nabla^L, 0)$ on the bundle gerbe (P, L, t).

If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, the corresponding trivialization of $\widehat{K}$ defines a unitary section σ of Γ, with $\delta\sigma = \tau$ and $\frac{1}{2\pi i}\nabla^\Gamma \sigma = \langle \mu, \theta^L \rangle \sigma$, where θ^L is the left-invariant Maurer-Cartan form on K. Thus $\mathcal{L} = (E, s)$, with E the trivial line bundle and $s = f_1^*\sigma$, is a pseudo-line bundle for $\mathcal{G}$. Given a principal connection θ, let ∇^E be the connection on the trivial bundle E, having connection 1-form $\langle \mu, \theta \rangle \in \Omega^1(P)$. Since $\frac{1}{2\pi i}\nabla^L s = f_1^* \langle \mu, \theta^L \rangle s$, it follows that

$$\frac{1}{2\pi i}((\delta\nabla^E)^{-1}\nabla^L)s = \langle \mu, f_1^*\theta^L - \delta\theta \rangle. \tag{3.3}$$

One finds $\partial_1^*\theta = \mathrm{Ad}_{f_1^{-1}}(\partial_0^*\theta - f_1^*\theta^L)$. Since μ is K-invariant, this shows that the right hand side of (3.3) vanishes. Thus ∇^E is a pseudo-line bundle connection. The error 2-form ω is given by

$$\pi^*\omega = \mathrm{d}\langle \mu, \theta \rangle = \langle \mu, \mathrm{d}\theta \rangle = \langle \mu, F^\theta \rangle.$$

All of these constructions can be made equivariant in a rather obvious way: Thus if G is another Lie group and P is a G-invariant principal K-bundle, any $\varrho \in \mathrm{Hom}(\pi_1(K), \mathrm{U}(1))$ defines a G-equivariant bundle gerbe (P, L, t) (with flat connection) over M. If ϱ is in the image of $\mu \in (\mathfrak{k}^*)^K$, there is a G-equivariant pseudo-line bundle for this gerbe. Furthermore any choice of G-equivariant principal connection on P defines a G-equivariant pseudo-line bundle connection, with equivariant error 2-form $\pi^*\omega_G = \langle \mu, F_G^\theta \rangle$ where $F_G^\theta \in \Omega_G^2(P, \mathfrak{k})$ is the equivariant curvature.

4. Gluing data

In this Section we describe a procedure for gluing a collection of bundle gerbes (X_i, L_i, t_i) on open subsets $V_i \subset M$, with pseudo-line bundles of their quotients on overlaps[2]. We begin with the somewhat simpler case that the surjective submersions $X_i \to V_i$ are obtained by restricting a surjective submersion $X \to M$, and later reduce the general case to this special case.

Thus, let $\pi \colon X \to M$ be a surjective submersion and let V_i, $i = 0, \ldots, d$ an open cover of M. Let $X_i = X|_{V_i}$, and more generally $X_I = X|_{V_I}$ where V_I is the intersection of all V_i with $i \in I$.

Suppose we are given bundle gerbes (X_i, L_i, t_i) over V_i and pseudo-line bundles (E_{ij}, s_{ij}) for the quotients $(X_{ij}, L_j L_i^{-1}, t_j t_i^{-1})$ over $V_i \cap V_j$, where $E_{ij} = E_{ji}^{-1}$ and $s_{ij} = s_{ji}^{-1}$. Note that $E_{ij} E_{jk} E_{ki}$ is a pseudo-line bundle for the trivial gerbe, hence is a pull-back $\pi^* F_{ijk}$ of a line bundle $F_{ijk} \to M$, and we will also require a unitary section u_{ijk} of that line bundle. Under suitable conditions the data (E_{ij}, s_{ij}) and u_{ijk} can be used to 'glue' the gerbes (X_i, L_i, t_i). The glued gerbe will be defined over the disjoint union $\coprod_{i=1}^d X_i$. We have

$$\Big(\coprod_{i=1}^{d} X_i\Big)^{[2]} = \coprod_{ij} X_i \times_M X_j$$

$$\Big(\coprod_{i=1}^{d} X_i\Big)^{[3]} = \coprod_{ijk} X_i \times_M X_j \times_M X_k$$

$$\ldots$$

Hence, the glued gerbe will be of the form $(\coprod_i X_i, \coprod_{ij} L_{ij}, \coprod_{ijk} t_{ijk})$ where L_{ij} are line bundles over $X_i \times_M X_j$ and t_{ijk} unitary sections of a line bundle $(\delta L)_{ijk}$

[2]) See Stevenson [29] for similar gluing constructions.