

## 2.3 Simplicial gerbes

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over  $X^{[3]}$ , satisfying a compatibility condition  $\delta t = 1$  over  $X^{[4]}$  (which makes sense since  $\delta t$  is a section of the canonically trivial bundle  $\delta\delta L$ ). Given local sections  $\sigma_a: U_a \rightarrow X$ , one can pull these data back under the maps  $(\sigma_a, \sigma_b): U_a \cap U_b \rightarrow X^{[2]}$  and  $(\sigma_a, \sigma_b, \sigma_c): U_a \cap U_b \cap U_c \rightarrow X^{[3]}$  to obtain a Chatterjee-Hitchin gerbe. The Dixmier-Douady class of  $(X, L, t)$  is by definition the Dixmier-Douady class of this Chatterjee-Hitchin gerbe; again this is independent of all choices. The Dixmier-Douady class behaves naturally under tensor product, pull-back and duals.

Notice that Chatterjee-Hitchin gerbes may be viewed as a special case of bundle gerbes, with  $X$  the disjoint union of the sets  $U_a$  in the given cover.

REMARK 2.1. In his original paper [24] Murray considered bundle gerbes only for fiber bundles, but this was found too restrictive. In [25], [29] the weaker condition (called ‘locally split’) is used that every point  $x \in M$  admits an open neighborhood  $U$  and a map  $\sigma: U \rightarrow X$  such that  $\pi \circ \sigma = \text{id}$ . However, this condition seems insufficient in the smooth category, as the fiber product  $X \times_M X$  need not be a manifold unless  $\pi$  is a submersion.

### 2.3 SIMPLICIAL GERBES

Murray’s construction fits naturally into a wider context of *simplicial gerbes*. We refer to Mostow-Perchik’s notes of lectures by R. Bott [23] and to Dupont’s paper [12] for a nice introduction to simplicial manifolds, and to Stevenson [29] for their appearance in the gerbe context.

Recall that a *simplicial manifold*  $M_\bullet$  is a sequence of manifolds  $(M_n)_{n=0}^\infty$ , together with *face maps*  $\partial_i: M_n \rightarrow M_{n-1}$  for  $i = 0, \dots, n$  satisfying relations  $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$  for  $i < j$ . (The standard definition also involves *degeneracy maps* but these need not concern us here.) The *(fat) geometric realization* of  $M_\bullet$  is the topological space  $\|M\| = \coprod_{n=1}^\infty \Delta^n \times M_n / \sim$ , where  $\Delta^n$  is the  $n$ -simplex and the relation is  $(t, \partial_i(x)) \sim (\partial^i(t), x)$ , for  $\partial^i: \Delta^{n-1} \rightarrow \Delta^n$  the inclusion as the  $i$ th face. A (smooth) simplicial map between simplicial manifolds  $M_\bullet, M'_\bullet$  is a collection of smooth maps  $f_n: M_n \rightarrow M'_n$  intertwining the face maps; such a map induces a map between the geometric realizations.

#### EXAMPLES 2.2.

(a) If  $S$  is any manifold, one can define a simplicial manifold  $E_\bullet S$  where  $E_n S$  is the  $n+1$ -fold cartesian product of  $S$ , and  $\partial_j$  omits the  $j$ th factor. It is known [23] that the geometric realization  $\|ES\|$  of this simplicial manifold is contractible. More generally, if  $X \rightarrow M$  is a fiber bundle with fiber  $S$ ,

one can define a simplicial manifold  $E_n X := X^{[n+1]}$ , with face maps as in Section 2.2. The geometric realization  $\|EX\|$  becomes a fiber bundle over  $M$  with contractible fiber  $\|ES\|$ .

(b) [22, 27] For any Lie group  $G$  there is a simplicial manifold  $B_n G = G^n$ . The face maps  $\partial_i$  for  $0 < i < n$  are

$$\partial_i(g_1, \dots, g_n) = (g_1, \dots, g_i g_{i+1}, \dots, g_n),$$

while  $\partial_0$  omits the first component and  $\partial_n$  the last component. The map  $\pi_n: E_n G \rightarrow B_n G$  given by  $\pi_n(k_0, \dots, k_n) = (k_0 k_1^{-1}, \dots, k_{n-1} k_n^{-1})$  is simplicial, and the induced map on geometric realizations is a model for the classifying bundle  $EG \rightarrow BG$ .

(c) [27, 23] If  $\mathcal{U} = \{U_a, a \in A\}$  is an open cover of  $M$ , one defines a simplicial manifold

$$\mathcal{U}_n M := \coprod_{(a_0, \dots, a_n) \in A_n} U_{a_0 \dots a_n}$$

where  $A_n$  is the set of all sequences  $(a_0, \dots, a_n)$  such that  $U_{a_0 \dots a_n} := U_{a_0} \cap \dots \cap U_{a_n}$  is non-empty. The face maps are induced by the inclusions,

$$\partial_i: U_{a_0 \dots a_n} \hookrightarrow U_{a_0 \dots \widehat{a_i} \dots a_n}.$$

One may view this as a special case of (a), with  $X = \coprod_{a \in A} U_a$ . It is known [23, Theorem 7.3] that  $\|\mathcal{U}M\|$  is homotopy equivalent to  $M$ .

(d) [2] The definitions of  $E_n G$  and  $B_n G$  extend to Lie groupoids  $G$  over a base  $S$ . If  $s, t: G \rightarrow S$  are the source and target maps, one defines  $E_n G$  as the  $n+1$ -fold fiber product of  $G$  with respect to the target map  $t$ . The space  $B_n G$  for  $n \geq 1$  is the set of all  $(g_1, \dots, g_n) \in G^n$  with  $s(g_j) = t(g_{j-1})$ , while  $B_0 G = S$ . The definition of the face maps  $\partial_j: B_n G \rightarrow B_{n-1} G$  is as before for  $n > 1$ , while for  $n = 1$ ,  $\partial_0 = t$  and  $\partial_1 = s$ . We have a simplicial map  $E_n G \rightarrow B_n G$  defined just as in the group case.  $\cdot$

The bi-graded space of differential forms  $\Omega^\bullet(M_\bullet)$  carries two commuting differentials  $d, \delta$ , where  $d$  is the de Rham differential and  $\delta: \Omega^k(M_n) \rightarrow \Omega^k(M_{n+1})$  is an alternating sum,  $\delta\alpha = \sum_{i=0}^{n+1} (-1)^i \partial_i^* \alpha$ . It is known [23, Theorem 4.2, Theorem 4.5] that the total cohomology of this double complex is the (singular) cohomology of the geometric realization, with coefficients in  $\mathbf{R}$ .

We will use the  $\delta$  notation in many similar situations: For instance, given a Hermitian line bundle  $L \rightarrow M_n$ , we define a Hermitian line bundle  $\delta L \rightarrow M_{n+1}$  as a tensor product,

$$\delta L = \partial_0^* L \otimes \partial_1^* L^{-1} \otimes \cdots \otimes \partial_{n+1}^* L^{\pm}.$$

The line bundle  $\delta(\delta L) \rightarrow M_{n+1}$  is canonically trivial, due to the relations between face maps. If  $\sigma$  is a unitary section (i.e. a trivialization) of  $L$ , one uses a similar formula to define a unitary section  $\delta\sigma$  of  $\delta L$ . Then  $\delta(\delta\sigma) = 1$  (the identity section of the trivial line bundle  $\delta(\delta L)$ ). For any unitary connection  $\nabla$  of  $L$ , one defines a unitary connection  $\delta\nabla$  of  $\delta L$  in the obvious way.

CONVENTION. For the rest of this paper, we take all line bundles  $L$  to be *Hermitian* line bundles, and all connections  $\nabla$  on  $L$  to be *unitary* connections.

Let  $M_{\bullet}$  be a simplicial manifold. One might define a simplicial line bundle as a collection of line bundles  $L_n \rightarrow M_n$  such that the face maps  $\partial_i: M_n \rightarrow M_{n-1}$  lift to line bundle homomorphisms  $\hat{\partial}_i: L_n \rightarrow L_{n-1}$ , satisfying the face map relations. Thus  $L_{\bullet}$  is itself a simplicial manifold, and its geometric realization  $\|L\|$  is a line bundle over  $\|M\|$ . Equivalently, the lifts  $\hat{\partial}_i$  may be viewed as isomorphisms,  $\partial_i^* L_{n-1} \rightarrow L_n$ . In particular, we may identify  $L_n$  with the pull-back of  $L := L_0$  under the  $n$ th-fold iterate  $\partial_0 \circ \cdots \circ \partial_0$ .

The isomorphisms  $\partial_1^* L \cong \partial_0^* L = L_1$  determine a unitary section  $t$  of  $\delta L \rightarrow M_1$ , and the compatibility of isomorphisms

$$(\partial_0 \partial_2)^* L \cong (\partial_0 \partial_1)^* L \cong (\partial_0 \partial_0)^* L = L_2$$

amount to the condition  $\delta t = 1$ . (Compatibility of the isomorphisms for  $L_n$  with  $n \geq 3$  is then automatic.) That is, a *simplicial line bundle over  $M_{\bullet}$*  is given by a line bundle  $L \rightarrow M_0$ , together with a unitary section  $t$  of  $\delta L \rightarrow M_1$ , such that  $\delta t = 1$  over  $M_2$ . A unitary section  $s$  of  $L$  with  $\delta s = t$  induces a unitary section of  $\|L\| \rightarrow \|M\|$ .

Taking  $L$  to be trivial, we see in particular that any  $U(1)$ -valued function  $t$  on  $M_1$ , with  $\delta t = 1$ , defines a line bundle over the geometric realization. A trivialization of that line bundle is given by a  $U(1)$ -valued function on  $M_0$  satisfying  $\delta s = t$ . Replacing  $U(1)$ -valued functions with line bundles, this motivates the following definition.

DEFINITION 2.3. A *simplicial gerbe over  $M_{\bullet}$*  is a pair  $(L, t)$ , consisting of a line bundle  $L \rightarrow M_1$ , together with a section  $t$  of  $\delta L \rightarrow M_2$  satisfying  $\delta t = 1$ . A *pseudo-line bundle for  $(L, t)$*  is a pair  $(E, s)$ , consisting of a line bundle  $E \rightarrow M_0$  and a section  $s$  of  $\delta E^{-1} \otimes L$  such that  $\delta s = t$ .

## REMARK 2.4.

(a) We are using the notion of a simplicial gerbe only as a 'working definition'. It is clear from the discussion above that a more general notion would involve a gerbe over  $M_0$ .

(b) In [9], what we call simplicial gerbe is called a simplicial line bundle. The name pseudo-line bundle is adopted from [9], where it is used in a similar context.

A simplicial gerbe over  $\mathcal{U} \bullet M$  (for a cover  $\mathcal{U}$  of  $M$ ) is a Chatterjee-Hitchin gerbe, while a simplicial gerbe over  $E \bullet X = X^{[\bullet+1]}$  (for a surjective submersion  $X \rightarrow M$ ) is a bundle gerbe. It is shown in [24] that the characteristic class of a bundle gerbe  $(X, L, t)$  vanishes if and only if it admits a pseudo-line bundle.

EXAMPLE 2.5 (Central extensions). (See [9, p.615].) Let  $K$  be a Lie group. A simplicial line bundle over  $B \bullet K$  is the same thing as a group homomorphism  $K \rightarrow \mathrm{U}(1)$ : The line bundle  $L \rightarrow B_0 K$  is trivial since  $B_0 K$  is just a point, hence the unitary section  $t$  of  $\delta L$  becomes a  $\mathrm{U}(1)$ -valued function. The condition  $\delta t = 1$  means that this function is a group homomorphism.

Similarly, a simplicial gerbe  $(\Gamma, \tau)$  over  $B \bullet K$  is the same thing as a central extension

$$\mathrm{U}(1) \rightarrow \widehat{K} \rightarrow K.$$

Indeed, given the line bundle  $\Gamma \rightarrow K$  let  $\widehat{K}$  be the unit circle bundle inside  $\Gamma$ . The fiber of  $\delta\Gamma \rightarrow K^2$  at  $(k_1, k_2)$  is a tensor product  $\Gamma_{k_2} \Gamma_{k_1 k_2}^{-1} \Gamma_{k_1}$ , hence the section  $\tau$  of  $\delta\Gamma \rightarrow K^2$  defines a unitary isomorphism  $\Gamma_{k_1} \Gamma_{k_2} \cong \Gamma_{k_1 k_2}$ , or equivalently a product on  $\widehat{K}$  covering the group multiplication on  $K$ . Finally, the condition  $\delta\tau = 1$  is equivalent to associativity of this product.

A pseudo-line bundle  $(E, s)$  for the simplicial gerbe  $(\Gamma, \tau)$  is the same thing as a splitting of the central extension: Obviously  $E$  is trivial since  $B_0 K$  is just a point; the section  $s$  defines a trivialization  $\widehat{K} = K \times \mathrm{U}(1)$ , and  $\delta s = t$  means that this is a group homomorphism.

DEFINITION 2.6. A connection on a simplicial gerbe  $(L, t)$  over  $M \bullet$  is a line bundle connection  $\nabla^L$ , together with a 2-form  $B \in \Omega^2(M_0)$ , such that  $(\delta\nabla^L)t = 0$  and

$$\delta B = \frac{1}{2\pi i} \mathrm{curv}(\nabla^L).$$

Given a pseudo-line bundle  $\mathcal{L} = (E, s)$ , we say that  $\nabla^E$  is a pseudo-line bundle connection if it has the property  $((\delta\nabla^E)^{-1}\nabla^L)s = 0$ .

Simplicial gerbes need not admit connections in general. A sufficient condition for the existence of a connection is that the  $\delta$ -cohomology of the double complex  $\Omega^k(M_n)$  vanishes in bidegrees  $(1, 2)$  and  $(2, 1)$ . In particular, this holds true for bundle gerbes: Indeed it is shown in [24] that for any surjective submersion  $\pi: X \rightarrow M$  the sequence

$$(2.1) \quad 0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(X) \xrightarrow{\delta} \Omega^k(X^{[2]}) \xrightarrow{\delta} \Omega^k(X^{[3]}) \xrightarrow{\delta} \dots$$

is exact, so the  $\delta$ -cohomology vanishes in *all* degrees.

Thus, every bundle gerbe  $\mathcal{G} = (X, L, t)$  over a manifold  $M$  (and in particular every Chatterjee-Hitchin gerbe) admits a connection. One defines the *3-curvature*  $\eta \in \Omega^3(M)$  of the bundle gerbe connection by  $\pi^*\eta = dB \in \ker \delta$ . It can be shown that its cohomology class is the image of the Dixmier-Douady class  $[\mathcal{G}]$  under the map  $H^3(M, \mathbf{Z}) \rightarrow H^3(M, \mathbf{R})$ . Similarly, if  $\mathcal{G}$  admits a pseudo-line bundle  $\mathcal{L} = (E, s)$ , one can always choose a pseudo-line bundle connection  $\nabla^E$ . The difference  $\frac{1}{2\pi i} \text{curv}(\nabla^E) - B$  is  $\delta$ -closed and one defines the *error 2-form* of this connection by

$$\pi^*\omega = \frac{1}{2\pi i} \text{curv}(\nabla^E) - B.$$

It is clear from the definition that  $d\omega + \eta = 0$ .

REMARK 2.7. There is a notion of holonomy around surfaces for gerbe connections (cf. Hitchin [18] and Murray [24]), and in fact gerbe connections can be defined in terms of their holonomy (see Mackaay-Picken [20]).

## 2.4 EQUIVARIANT BUNDLE GERBES

Suppose  $G$  is a Lie group acting on  $X$  and on  $M$ , and that  $\pi: X \rightarrow M$  is a  $G$ -equivariant surjective submersion. Then  $G$  acts on all fiber products  $X^{[p]}$ . We will say that a bundle gerbe  $\mathcal{G} = (X, L, t)$  is *G-equivariant*, if  $L$  is a  $G$ -equivariant line bundle and  $t$  is a  $G$ -invariant section. An equivariant bundle gerbe defines a gerbe over the Borel construction<sup>1)</sup>  $X_G = EG \times_G X \rightarrow M_G = EG \times_G M$ , hence has an *equivariant* Dixmier-Douady class in  $H^3(M_G, \mathbf{Z}) = H_G^3(M, \mathbf{Z})$ . Similarly, we say that a pseudo-line bundle  $(E, s)$  for  $(X, L, t)$  is equivariant, provided  $E$  carries a  $G$ -action and  $s$  is an invariant section.

<sup>1)</sup> We have not discussed bundle gerbes over infinite-dimensional spaces such as  $M_G$ . Recall however [4] that the classifying bundle  $EG \rightarrow BG$  may be approximated by finite-dimensional principal bundles, and that equivariant cohomology groups of a given degree may be computed using such finite dimensional approximations.