

12. Back to mapping-telescopes

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Proposition 10.1 provides a $\kappa(r, s) = Bi(A(r, s), A(r, s))$ such that this bigon is $\kappa(r, s)$ -thin. Thus the given (r, s) -chain bigon is $\delta(r, s)$ -thin, with $\delta(r, s) = \kappa(r, s) + 2A(r, s)$. By Lemma 11.1, the given forest-stack, which is a $(1, 2)$ -quasi geodesic metric space, is $2\delta(1, 6)$ -hyperbolic. $\square$

12. BACK TO MAPPING-TELESCOPES

In this section we elucidate the relationships between forest-stacks and mapping-telescopes.

12.1 STATEMENT OF THE THEOREM

An **R-tree** (see [9], [2] among many others) is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an **R-tree** is a topological tree. An **R-forest** is a union of disjoint **R-trees**.

LEMMA 12.1. *Let (Γ, d_Γ) be an **R-forest** and let $\psi: \Gamma \rightarrow \Gamma$ be a forest-map of Γ . Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) equipped with a structure of forest-stack as defined in Section 2. Then there is a horizontal metric $\mathcal{H} = (m_r)_{r \in \mathbf{R}}$ on K_ψ such that*

1. *The **R-forests** $(f^{-1}(r), m_r)$ and $(f^{-1}(r+1), m_{r+1})$ are isometric. Each stratum $(f^{-1}(n), m_n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ) .*
2. *For any real r and any horizontal geodesic $g \in f^{-1}(r)$, the map*

$$l_{r,g}: \begin{cases} +1 - r] \rightarrow \mathbf{R}^+ \\ t \mapsto |\sigma_t(g)|_{r+t} \end{cases}.$$

is monotone.

Such a horizontal metric is called a horizontal d_Γ -metric. The telescopic metric associated to a horizontal d_Γ -metric is called a mapping-telescope d_Γ -metric.

Proof. We make each $\Gamma \times \{n\}$, $n \in \mathbf{Z}$, an **R-forest** isometric to Γ . We consider a cover of Γ by geodesics of length 1 which intersect only at their endpoints. Each $\Gamma \times \{n\}$ inherits the same cover. There is a disc $D_{e,n}$ in K_ψ for each such horizontal geodesic e in $\Gamma \times \{n\}$. This disc is bounded by e , $\psi(e)$ and the orbit-segments between the endpoints of e and those of $\psi(e)$.

We foliate this disc by segments with endpoints in, and transverse to, the orbit-segments in its boundary. Then we assign a length to each such segment so that the collection of lengths varies continuously and monotonically, from the length of e to that of $\psi(e)$. We thus obtain a horizontal metric on the mapping-telescope. Furthermore each stratum $f^{-1}(n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ) . And the maps denoted by $l_{r,g}$ in Lemma 12.1 are monotone by construction. By definition of a mapping-telescope, the discs $D_{e,n}$ between $\Gamma \times \{n\}$ and $\Gamma \times \{n+1\}$ are copies of the discs $D_{e,n'}$ between $\Gamma \times \{n'\}$ and $\Gamma \times \{n'+1\}$, for any n, n' in $\mathbf{Z}$. This allows us to choose the horizontal metric to satisfy the further condition that $(f^{-1}(r), m_r)$ be isometric with $(f^{-1}(r+1), m_{r+1})$ for any real number r . $\square$

We now define dynamical properties for **R**-forest maps.

DEFINITION 12.2. Let (Γ, d_Γ) be an **R**-forest. A forest-map ψ of (Γ, d_Γ) is *weakly bi-Lipschitz* if there exist $\mu \geq 1$ and $K \geq 0$ such that $\mu d_\Gamma(x, y) \geq d_\Gamma(\psi(x), \psi(y)) \geq \frac{1}{\mu} d_\Gamma(x, y) - K$.

DEFINITION 12.3. Let (Γ, d_Γ) be an **R**-forest. A forest-map ψ of (Γ, d_Γ) is *hyperbolic* if it is weakly bi-Lipschitz and there exist $\lambda > 1$, $N \geq 1$, $M \geq 0$ such that for any pair of points x, y in Γ with $d_\Gamma(x, y) \geq M$, either $d_\Gamma(\psi^N(x), \psi^N(y)) \geq \lambda d_\Gamma(x, y)$ or $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$ for some x_N, y_N with $\psi^N(x_N) = x$, $\psi^N(y_N) = y$.

A hyperbolic forest-map ψ of (Γ, d_Γ) is *strongly hyperbolic* if, for any pair of points x, y with $d_\Gamma(x, y) \geq M$ and each connected component containing both a preimage of x and a preimage of y under ψ^N , there is at least one pair of such preimages x_N, y_N for which $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$.

If the forest Γ is a tree then a hyperbolic forest-map is strongly hyperbolic (similarly we saw that a hyperbolic semi-flow on a forest-stack whose strata are connected is strongly hyperbolic).

Our theorem about mapping-telescopes is

THEOREM 12.4. Let (Γ, d_Γ) be an **R**-forest. Let ψ be a strongly hyperbolic forest-map of (Γ, d_Γ) whose mapping-telescope K_ψ is connected. Then K_ψ is a Gromov-hyperbolic metric space for any mapping-telescope d_Γ -metric.

12.2 PROOF OF THEOREM 12.4

LEMMA 12.5. *Let (Γ, d_Γ) be an $\mathbf{R}$ -forest. Let ψ be a weakly bi-Lipschitz forest-map of (Γ, d_Γ) . Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) , equipped with a structure of forest-stack as defined in Section 2. Then the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is a bounded-cancellation and bounded-dilatation semi-flow with respect to any horizontal d_Γ -metric (see Lemma 12.1).*

Proof. The horizontal metric $\mathcal{H}$ agrees with the metric d_Γ on all the strata $f^{-1}(n)$, $n \in \mathbf{Z}$ (see Lemma 12.1). Consider any horizontal geodesic g in the stratum $f^{-1}(0)$. If ψ is weakly bi-Lipschitz with constants μ_0 and K_0 , then for any integer $n \geq 0$, we have $|[g]_n|_n \geq \frac{1}{\mu_0^n} |g|_0 - K_0 \left(\frac{1}{\mu_0^{n-1}} + \frac{1}{\mu_0^{n-2}} + \dots + 1 \right)$. Since $0 < \frac{1}{\mu_0} < 1$, the sum tends to $\frac{\mu_0}{\mu_0 - 1}$ as $n \rightarrow +\infty$. Setting $\lambda_- = \frac{1}{\mu_0}$ and $K = K_0 \frac{\mu_0}{\mu_0 - 1}$, this proves the inequality of item (1) for horizontal geodesics in $f^{-1}(n)$, $n \in \mathbf{Z}$, and an integer time t . For the case in which t is any positive real number and $g \in f^{-1}(r)$, r any real number, just decompose $\sigma_t = \sigma_{t-E[t]} \circ \sigma_{E[t-(E[r]+1-r)]} \circ \sigma_{E[r]+1-r}$. The map σ_t is a homeomorphism from $f^{-1}(r)$ onto $f^{-1}(r+t)$ for any $t \in [0, E[r]+1-r)$. That is, for any real r , $|[g]_{r+t}|_{r+t} = |\sigma_t(g)|_{r+t}$ for $t \in [0, E[r]+1-r)$. The monotonicity of the maps $l_{r,g}$ (see Lemma 12.1, item (2)) implies, for any r and $t \in [0, E[r]+1-r)$, that $|\sigma_t(g)|_{r+t} \geq \frac{1}{\mu_0} |g|_r$. The conclusion follows. $\square$

LEMMA 12.6. *With the assumptions and notation of Lemma 12.5, if the map ψ is a (strongly) hyperbolic forest-map of (Γ, d_Γ) then the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is (strongly) hyperbolic with respect to any horizontal d_Γ -metric.*

The proof is similar to that of Lemma 12.5. $\square$

Proof of Theorem 12.4. By Lemmas 12.5 and 12.6, a mapping-telescope admits a structure of forest-stack $(\tilde{X}, f, \sigma_t, \mathcal{H})$ with horizontal metric $\mathcal{H}$ such that the semi-flow $(\sigma_t)_{t \in \mathbf{R}^+}$ is a strongly hyperbolic semi-flow with respect to $\mathcal{H}$. Hence Theorem 4.4 implies Theorem 12.4. $\square$

13. ABOUT MAPPING-TORUS GROUPS

We first recall the definition of a *hyperbolic endomorphism* of a group introduced by Gromov [19].