1. ON THE TOPOLOGY OF A QUADRIC IN P_C^n

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1. On the topology of a quadric in $P^n_{\mathbf{C}}$

Let Q be a codimension 1, non-singular complex quadric in the projective space $P^n_{\mathbf{C}}$.

THEOREM 1.1. *The complement of Q in $P^n_{\mathbf{C}}$ is diffeomorphic to the total space of the tangent bundle of the n-dimensional real projective space:*

$$P^n_{\mathbf{C}} \setminus Q \cong T(P^n_{\mathbf{R}}) .$$

Proof. We first notice that a non-singular hypersurface of degree d in $P^n_{\mathbf{C}}$ is determined by a homogeneous polynomial of degree d in $n+1$ complex variables, with no critical points outside $0 \in \mathbf{C}^{n+1}$. Let $\mathcal{P}$ be the projective space of coefficients of homogeneous polynomials of degree d in $n+1$ complex variables. The general homogeneous equation of degree d in $n+1$ variables is

$$\sum_{\alpha_0+\cdots+\alpha_n=d} a_{\alpha_0,\ldots,\alpha_n} z_0^{\alpha_0} \ldots z_n^{\alpha_n} = 0 .$$

This defines a polynomial, and hence a hypersurface $\mathcal{X}$, in $\mathcal{P} \times P^n_{\mathbf{C}}$. The family of projective hypersurfaces of degree d in $P^n_{\mathbf{C}}$ is given by the map

$$\mathcal{E} \colon \mathcal{X} \to \mathcal{P} ,$$

induced by the projection of $\mathcal{P} \times P^n_{\mathbf{C}}$ onto $\mathcal{P}$. In $\mathcal{P}$, the polynomials defining singular hypersurfaces in $P^n_{\mathbf{C}}$ form a closed subvariety of complex codimension one. Hence its complement Ω is connected. Since the map $\mathcal{E}$ is a locally trivial fibration over Ω, by Ehresmann's lemma, one knows that any non-singular hypersurface $X \subset P^n_{\mathbf{C}}$ of degree d is ambient isotopic to the hypersurface defined by the Fermat polynomial $\mathbf{F}^n_d := z_0^d + \cdots + z_n^d$. That is, up to isotopy we can assume that X is the projectivization of the affine variety $V := \{z_0^d + \cdots + z_n^d = 0\}$ after removing the singular point $0 \in V$ (cf. [LC; Lemme 2.2]).

The projective space $P^n_{\mathbf{C}}$ is obtained dividing $\mathbf{C}^{n+1} - \{0\}$ by the $\mathbf{C}^*$-action:

$$g_t(z_0, \ldots, z_n) = (e^{it} z_0, \ldots, e^{it} z_n) , \qquad t \in \mathbf{C}^* = \mathbf{C} \setminus \{0\} .$$

Since V is an invariant set for this $\mathbf{C}^*$-action, it follows that $P^n_{\mathbf{C}} \setminus X$ is the image of $\mathbf{C}^{n+1} \setminus V$. Moreover, $\mathbf{C}^*$ is $S^1 \times \mathbf{R}^+$ and if we divide $\mathbf{C}^{n+1} \setminus \{0\}$ by the $\mathbf{R}^+$-action we get the sphere S^{2n-1}. Thus $P^n_{\mathbf{C}} \setminus X$ is the quotient of $S^{2n-1} \setminus (V \cap S^{2n-1})$ by the corresponding S^1-action. By [Mi2], these S^1-orbits are transversal to the Milnor fibres of the polynomial $\mathbf{F}^n_d(z) = z_0^d + \cdots + z_n^d$, and their action on the fibres is given by the monodromy, which is cyclic of period d. Therefore the Milnor fibre F is a d-fold cyclic cover of $P^n_{\mathbf{C}} \setminus X$.

In the quadratic case $d = 2$, the Milnor fibre is diffeomorphic to the affine variety $z_0^2 + \cdots + z_n^2 = 1$. Let us decompose each vector $Z := (z_0, \dots, z_n)$ into its real and imaginary parts, $Z = U + iV$; then the Milnor fibre is given as the set $(U, V) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ such that $|U|^2 - |V|^2 = 1$ and $U \perp V$. We notice that the map $(U, V) \mapsto (U/\|U\|, V)$ induces an isomorphism of this Milnor fibre with the tangent bundle of S^n. The monodromy is given by multiplication by -1, $(U, V) \mapsto (-U, -V)$. The quotient of F by this involution is, therefore, the tangent bundle of the real projective n-space. $\square$

We notice that part of the argument above is similar to that of Lemmas 2.2 and 2.3 in [LC] (see also Libgober in [Li; Lemma 1.1]), implying Corollary 1.2 below. We denote by X_0 the projectivization of the affine hypersurface defined by the Fermat polynomial $\mathbf{F}^n_d$, and by $C^n_d := P^n_{\mathbf{C}} \setminus X_0$ the complement of X_0.

COROLLARY 1.2. *Let X be a non-singular hypersurface of $P^n_{\mathbf{C}}$ of degree d. Then:*

i) *the pair $(P^n_{\mathbf{C}}, X)$ is isotopic to the pair $(P^n_{\mathbf{C}}, X_0)$; and*

ii) *the Milnor fibre F of $\mathbf{F}^n_d$ is a d-fold cyclic cover of C^n_d, the projection map $F \to C^n_d$ being given by the monodromy of the Milnor fibration of $\mathbf{F}^n_d$ (which is cyclic of period d).*

Since the Milnor fibre has the homotopy type of a bouquet of μ spheres S^n, by [Ph, Mi2], one has (as in [Li]) that for $n > 1$, the fundamental group $\pi_1(C^n_d)$ is isomorphic to $\mathbf{Z}/d\mathbf{Z}$, and $\pi_j(C^n_d) \cong \pi_j(\bigvee_\mu S^n)$, for $j > 1$, where $\mu = (d-1)^{n+1}$ is the Milnor number and $\bigvee_\mu S^n$ is a bouquet of μ spheres of dimension n. In particular:

$$\pi_j(C^n_d) = 0 \text{ if } 1 < j < n\,, \quad \text{and} \quad \pi_j(C^n_d) \cong \mathbf{Z}^\mu \text{ if } j = n\,. \tag{1.3}$$

We now let $Q = Q_{n-1} \subset P^n_{\mathbf{C}}$ be the non-singular hyperquadric in $P^n_{\mathbf{C}}$ with equation

$$z_0^2 + \cdots + z_n^2 = 0\,,$$

in homogeneous projective coordinates. Let $j\colon P^n_{\mathbf{C}} \to P^n_{\mathbf{C}}$ be the involution on $P^n_{\mathbf{C}}$ given by complex conjugation: $j([z_0, \dots, z_n]) = [\bar{z}_0, \dots, \bar{z}_n]$, and let Π be the fixed point set of j, so that $\Pi \cong P^n_{\mathbf{R}}$.

Theorem 1.1 says that $P^n_{\mathbf{C}} \setminus Q$ is diffeomorphic to the tangent bundle $T(\Pi)$, and Π is the zero section of this bundle. Hence $P^n_{\mathbf{C}} \setminus (Q \cup \Pi)$ can be regarded as the set of non zero tangent vectors of Π, so it is diffeomorphic to the cylinder $T_1(\Pi) \times (0, 1)$, where $T_1(\Pi)$ is the unit tangent bundle of Q. The group $\mathrm{SO}(n+1, \mathbf{R})$ acts linearly on $\mathbf{C}^{n+1}$ and this action descends to an action on $P^n_{\mathbf{C}}$ which preserves Q. This action also leaves invariant the real projective space Π, where it acts in the usual way (i.e. via the action induced from the linear $\mathrm{SO}(n+1, \mathbf{R})$-action on $\mathbf{R}^{n+1}$). This extends, via the differential, to a transitive action of $\mathrm{SO}(n+1, \mathbf{R})$ on $T_1(\Pi)$, with isotropy subgroup $\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}$. Hence $T_1(\Pi)$ is diffeomorphic to $\mathrm{SO}(n+1, \mathbf{R})/(\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z})$. But $\mathrm{SO}(n+1, \mathbf{R})$ also acts transitively on $F^{n+1}_{+}(2, 1)$, the (partial) flag manifold of *oriented* 2-planes in $\mathbf{R}^{n+1}$ and (non-oriented) lines in these planes, with isotropy $\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}$. Thus one has diffeomorphisms

$$T_1(\Pi) \cong \mathrm{SO}(n+1, \mathbf{R})/(\mathrm{SO}(n-1, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}) \cong F^{n+1}_{+}(2, 1).$$

The Milnor fibre of the Fermat quadric $\mathbf{F}^n_2 = 0$ in $\mathbf{C}^{n+1}$ is diffeomorphic to the total space of the tangent bundle TS^n. Thus the link K of this singularity is diffeomorphic to the unit tangent bundle of S^n. Hence K is diffeomorphic to the Stiefel manifold $V_{n+1,2}$ of orthonormal 2-frames in $\mathbf{R}^{n+1}$. Therefore $Q \subset P^n_{\mathbf{C}}$, being the projectivization of K, is diffeomorphic to the Grassmannian $G_{n+1,2}$ of oriented 2-planes in $\mathbf{R}^{n+1}$. Thus one has a double fibration:

$$\begin{tikzcd} & F^{n+1}_{+}(2,1) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Q & & P^n_{\mathbf{R}} \end{tikzcd} \tag{1.4}$$

where π_1 and π_2 are the maps that assign to each flag (P, l) either the 2-plane $P \in G_{n+1,2}$ or the line $l \in P^n_{\mathbf{R}}$.

We form the corresponding double mapping cylinder $\big(F^{n+1}_{+}(2, 1) \times [0, 1]\big)/\sim$, where $\sim$ identifies a point

$$((P_0, l_0), 0) \in F^{n+1}_{+}(2, 1) \times \{0\}$$

with the point $\pi_1(P_0, l_0) = P_0$ in $G_{n+1,2} \cong Q$, and a point

$$((P_1, l_1), 1) \in F^{n+1}_{+}(2, 1) \times \{1\}$$

with the point $\pi_2(P_1, l_1) = l_1 \in P^n_{\mathbf{R}}$. The space we obtain is homeomorphic to $P^n_{\mathbf{C}}$. Furthermore, the double fibration (1.4) splits into two fibrations, corresponding to the maps π_1 and π_2. In the first case the space we get is the open mapping cylinder of π_1, and this is $P^n_{\mathbf{C}} \setminus \Pi$, while in the second case we get $P^n_{\mathbf{C}} \setminus Q$, which is the open mapping cylinder of π_2. One has the following

THEOREM 1.5. *The projective space $P^n_{\mathbf{C}}$ is the double mapping cylinder of the double fibration (1.4). If we remove Q from $P^n_{\mathbf{C}}$ we obtain a manifold diffeomorphic to the total space of the normal bundle of $\Pi \cong P^n_{\mathbf{R}}$ in $P^n_{\mathbf{C}}$. Reciprocally, if we remove Π from $P^n_{\mathbf{C}}$, what we get is diffeomorphic to the total space of the normal bundle of Q in $P^n_{\mathbf{C}}$. If we remove $Q \cup \Pi$ from $P^n_{\mathbf{C}}$, what we get is diffeomorphic to $F^{n+1}_{+}(2,1) \times (0,1)$, where*

$$F^{n+1}_{+}(2,1) \cong \mathrm{SO}(n+1,\mathbf{R})/\big(\mathrm{SO}(n-1,\mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}\big)$$

is the (partial) flag manifold of oriented 2-planes in $\mathbf{R}^{n+1}$ and (non-oriented) lines in these planes.

Proof. We notice that if we replace in Theorem (1.5) the word *diffeomorphic* by *homeomorphic*, then this theorem follows immediately from the previous discussion. Let us prove that we actually have diffeomorphisms. By Theorem 1.1, this is clear for $P^n_{\mathbf{C}} \setminus Q$. In fact, the fibration of $P^n_{\mathbf{C}} \setminus (Q \cup \Pi)$ given by the manifolds $F^{n+1}_{+}(2,1)$ corresponds to the fibration on $T(\Pi) \setminus \Pi$ given by sphere bundles of radius $r > 0$, for some metric on $T(\Pi)$. These correspond to boundaries of tubular neighbourhoods $\tilde{\nu}_r(\Pi)$ of $\Pi \subset P^n_{\mathbf{C}}$. In particular $P^n_{\mathbf{C}} \setminus Q$ is a tubular neighbourhood of Π, hence $P^n_{\mathbf{C}} \setminus Q$ is diffeomorphic to the total space of the normal bundle of $\Pi \cong P^n_{\mathbf{R}}$ in $P^n_{\mathbf{C}}$. This bundle is isomorphic to $T(\Pi)$.

Let us prove that $P^n_{\mathbf{C}} \setminus \Pi$ is diffeomorphic to the total space of the normal bundle of Q in $P^n_{\mathbf{C}}$. We observe that for all $r > 0$, the interior of $P^n_{\mathbf{C}} \setminus \tilde{\nu}_r(\Pi)$ is diffeomorphic to $P^n_{\mathbf{C}} \setminus \Pi$. Now we prove that $P^n_{\mathbf{C}} \setminus \Pi$ is actually a tubular neighbourhood of Q. For this we recall that if N is a Riemannian submanifold of $P^n_{\mathbf{C}}$, its *normal map* $\mathcal{N}_N$ is the function that associates, to each normal vector v of N in $P^n_{\mathbf{C}}$, the projection to $P^n_{\mathbf{C}}$ (via the exponential map) of the end-point of $v \in TP^n_{\mathbf{C}}$ (see, for instance [Mi1], p. 32, or [AG]). Let us denote by $\nu(Q)$ the normal bundle of Q in $P^n_{\mathbf{C}}$ and consider the normal map

$$\mathcal{N}_Q \colon \nu(Q) \to P^n_{\mathbf{C}} .$$

We notice that every complex projective line $\mathcal{L}$ in $P^n_{\mathbf{C}}$ orthogonal to Q, for the Fubini-Study metric, is invariant under conjugation, which is an isometry. So $\mathcal{L}$ is defined by equations with real coefficients (cf. §2 below), and it is totally geodesic in $P^n_{\mathbf{C}}$, since it is a complex projective line. Therefore $\mathcal{L}$ intersects Π transversally in a real projective line. This implies that the normal map $\mathcal{N}_Q$ is a diffeomorphism from the open disk bundle in $\nu(Q)$ of radius $\frac{\pi}{2}$ into $P^n_{\mathbf{C}} \setminus \Pi$. The union of all closed geodesic segments normal to Q of length $\frac{\pi}{2}$ fill up all of $P^n_{\mathbf{C}}$. Thus the distance from a point $p \in P^n_{\mathbf{C}} \setminus (\Pi \cup Q)$ to Q is exactly the length of the unique geodesic segment joining p and the unique point $q \in Q$ such that this segment is orthogonal to Q. Hence every tubular neighbourhood of Q in $P^n_{\mathbf{C}}$, of diameter less than $\frac{\pi}{2}$, is diffeomorphic to $P^n_{\mathbf{C}} \setminus \Pi$. $\square$

We remark that one has a construction for the Milnor fibre F of the Fermat polynomial $\mathbf{F}^n_2$ in the spirit of Theorem (1.5), since F can be regarded as the open mapping cylinder of the fibration

$$V_{n+1,2} \cong \mathrm{SO}(n+1,\mathbf{R})/\,\mathrm{SO}(n-1,\mathbf{R}) \longrightarrow \mathrm{SO}(n+1,\mathbf{R})/\,\mathrm{SO}(n,\mathbf{R}) \cong S^n\,,$$

where $V_{n+1,2}$ is the aforementioned Stiefel manifold.

2. On the geometry of $P^n_{\mathbf{C}}$

We now look more carefully at the decomposition of $P^n_{\mathbf{C}}$ arising from the double fibration (1.4). For this, it is convenient to look at two other interesting foliations that arise naturally from the double fibration (1.4), and from other considerations too.

The first foliation $\mathcal{F}_1$ is actually defined on $P^n_{\mathbf{C}} \setminus \Pi$ and its leaves are the fibres of π_1, which are 2-disks transversal to Q, by Theorem 1.5. By construction, each leaf of $\mathcal{F}_1$ is transversal to all the manifolds $F^{n+1}_+(2,1) \times t \subset P^n_{\mathbf{C}}$ for $t \in (0,1)$, intersecting each in a copy of $P^1_{\mathbf{R}}$ and approaching Π as $t \to 1$. Let us construct this foliation in a different way. We endow $P^n_{\mathbf{C}}$ with the Fubini-Study metric. From the proof of Theorem 1.5 we know that the normal map $\mathcal{N}_Q$ of Q induces a diffeomorphism between the open disk bundle of radius $\pi/2$ and $P^n_{\mathbf{C}} \setminus \Pi$. The leaves of $\mathcal{F}_1$ are the images of the normal disks. Since the conjugation $j\colon P^n_{\mathbf{C}} \to P^n_{\mathbf{C}}$ is an isometry, we have that a projective line $\mathcal{L}$ in $P^n_{\mathbf{C}}$ intersects Q at two conjugate points iff it is orthogonal to Q, and this happens iff $\mathcal{L}$ can be defined by equations with real coefficients. So we call these **CR**-lines. If two distinct **CR**-lines intersect, they do so in a point in $\Pi \cong P^n_{\mathbf{R}}$. Also, each **CR**-line $\mathcal{L}$ meets Π in a real projective line, which is an equator of $\mathcal{L}$. Since all complex lines in $P^n_{\mathbf{C}}$ are totally geodesic, the real projective line $\mathcal{L} \cap \Pi$ is a geodesic in $P^n_{\mathbf{C}}$, at equal distance $\pi/2$ from both intersection points in $\mathcal{L} \cap Q$. This divides $\mathcal{L}$ into two round disks