0. Introduction

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QUADRICS, ORTHOGONAL ACTIONS AND INVOLUTIONS IN COMPLEX PROJECTIVE SPACES

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0. INTRODUCTION

The purpose of this article is to look at the canonical action of the special orthogonal group $\mathrm{SO}(n+1,\mathbf{R})$ on $P^n_{\mathbf{C}}$, the complex projective space, in order to get a better understanding of the geometry and topology of the latter. This is related with a classical problem, studied by Zariski [Za] and others, of studying the topology of the complement of an affine algebraic hypersurface $V \subset \mathbf{C}^{n+1}$, in the particular case when V is a homogeneous quadric with an isolated singularity at the origin. We actually look at the projectivized situation. We begin by showing that the complement of a non-singular hyperquadric Q in $P^n_{\mathbf{C}}$ is diffeomorphic to the total space of the tangent bundle of the real projective n-space $P^n_{\mathbf{R}}$,

$$P^n_{\mathbf{C}} \setminus Q \cong T(P^n_{\mathbf{R}})\,.$$

For $n=3$, this implies that the complement of the nonsingular quadric in $P^3_{\mathbf{C}}$ is diffeomorphic to the group $P\mathrm{SL}(2,\mathbf{C})$. Then we use the above observation on the topology of $P^n_{\mathbf{C}} \setminus Q$ to describe $P^n_{\mathbf{C}}$ as the double mapping cylinder of the double fibration

$$\begin{tikzcd} & F^{n+1}_{+}(2,1) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Q & & P^n_{\mathbf{R}} \end{tikzcd}$$

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where

$$F_{+}^{n+1}(2,1) \cong \mathrm{SO}(n+1,\mathbf{R})/(\mathrm{SO}(n-1,\mathbf{R}) \times \mathbf{Z}/2\mathbf{Z})$$

is the partial flag manifold of *oriented* 2-planes in $\mathbf{R}^{n+1}$ and *non-oriented* lines in these planes. The manifold $F_{+}^{n+1}(2,1)$ is diffeomorphic to the unit sphere normal bundle of Q in $P_{\mathbf{C}}^{n}$, and it is also diffeomorphic to the unit sphere tangent bundle of $P_{\mathbf{R}}^{n}$.

V. Vassiliev pointed out to us that this decomposition of $P_{\mathbf{C}}^{n}$ resembles the one he gave in [Va1]. In fact, as we explain at the end of Section 3 below, in the case $n=2$ the above decomposition of $P_{\mathbf{C}}^{2}$ descends (by complex conjugation) to a similar decomposition of the 4-sphere. If we denote by $F^3(2,1) \cong F_{+}^{3}(2,1)\,/\,(\mathbf{Z}/2\mathbf{Z})$ the flag manifold of non-oriented 2-planes in $\mathbf{R}^3$ and non-oriented lines in these planes, then the sphere S^4 is obtained by taking the cylinder $F^3(2,1)\times[0,1]$ and gluing two copies of $P_{\mathbf{R}}^{2}$ to its boundary, via the obvious projections. This is explained (in different words) in Theorem 2 of [Va1], where he uses it to show that the flag manifold $F^3(2,1)$ is the Spanier-Whitehead dual (in the sense of [SW]) of two disjoint copies of $P_{\mathbf{R}}^{2}$. Also, in the remark at the end of that article, Vassiliev acknowledges an explanation by S. M. Finashin relating to $P_{\mathbf{C}}^{2}$ his construction on S^4. Thus, in the case $n=2$ our construction is actually hidden in Vassiliev's article. He explained to us that his method can also be used in higher dimensions to obtain our construction for $P_{\mathbf{C}}^{n}$ in general. It is interesting to observe, as pointed out to us by E. Ghys, that this decomposition of S^4 actually corresponds to the Tits building for the symmetric space $\mathrm{SL}(3,\mathbf{R})/\,\mathrm{SO}(3,\mathbf{R})$. This quotient can be regarded as the 5-ball, whose *visual sphere* at infinity is S^4. We refer to [BGS, Eb] for details on this construction (especially §9 in Appendix 5 of [BGS]). Our construction for $n=2$ is also related with the study done by C. T. C. Wall in [Wa] about Klein's formula for real projective plane curves.

In Section 2 we look more carefully at the decomposition of $P_{\mathbf{C}}^{n}$ arising from the above double fibration. This describes $P_{\mathbf{C}}^{n}$ as a 1-parameter family of codimension 1 submanifolds $F_{+}^{n+1}(2,1)\times\{t\}$, for $t\in(0,1)$, together with two "special" fibres: Q and a copy of the real projective space. We prove that these are the orbits of the natural action of $\mathrm{SO}(n+1,\mathbf{R})$ on $P_{\mathbf{C}}^{n}$, regarded as a subgroup of the complex orthogonal group $\mathrm{SO}(n+1,\mathbf{C})$. This is an isometric action, with respect to the Fubini-Study metric on $P_{\mathbf{C}}^{n}$, and the principal orbits are the flag manifolds $F_{+}^{n+1}(2,1)$, which have codimension 1. So this is an isometric action on $P_{\mathbf{C}}^{n}$ of cohomogeneity 1, thus it is hyperpolar, i.e. there is an embedded geodesic circle, transversal to all the orbits, by [HPTT]. Here we exhibit such a circle explicitly and we use it to parametrize the space of orbits, which is the interval $[0, \frac{\pi}{2}]$. The endpoints of this interval correspond to the two special orbits, which are the quadric Q and the real projective space Π which is the fixed point set of the complex conjugation in $P^n_{\mathbf{C}}$.

For this, it is convenient to look at two other interesting foliations which arise naturally from the double fibration (1.4), and from other considerations too. The first foliation $\mathcal{F}_1$ is defined on $P^n_{\mathbf{C}} \setminus \Pi$ and its leaves are open 2-disks transversal to Q and transversal to all the $\mathrm{SO}(n+1, \mathbf{R})$-orbits on $P^n_{\mathbf{C}} \setminus \Pi$. To construct this foliation, we let $\mathcal{N}$ be the normal map of Q. This map is defined on the normal bundle of Q in $P^n_{\mathbf{C}}$ and it is the restriction to this normal bundle of the exponential map. We show that this map is regular for normal vectors of length less than $\frac{\pi}{2}$ and it carries each normal sphere bundle of Q of radius less than $\frac{\pi}{2}$ into an $\mathrm{SO}(n+1, \mathbf{R})$-orbit. The image under $\mathcal{N}$ of the 2-disks orthogonal to Q are the leaves of the foliation $\mathcal{F}_1$ on $P^n_{\mathbf{C}} \setminus \Pi$. The space Π is the set of *focal points* of Q, i.e. the image under the exponential map of the set of critical values of the normal map. The closure of each leaf of $\mathcal{F}_1$ is a closed 2-disk that meets Π orthogonally in a projective line which is a closed geodesic in $P^n_{\mathbf{C}}$. For each pair of conjugate points in Q, the corresponding leaves are naturally glued together along their common limit set in Π, forming a complex projective line defined by real coefficients. The second foliation $\mathcal{F}_2$ is defined on $P^n_{\mathbf{C}} \setminus Q$; its leaves are embedded n-disks orthogonal to Π. These are the image under the normal map $\mathcal{N}$ of the fibres of the normal disk bundle of Π of radius less than $\frac{\pi}{2}$. The leaves are everywhere transversal to the orbits of $\mathrm{SO}(n+1, \mathbf{R})$. The quadric Q is the set of focal points of Π, and the closure of each leaf in $\mathcal{F}_2$ is a closed n-disk that meets Q orthogonally in a $(n-1)$-sphere, invariant under complex conjugation. The space Π is embedded in $P^n_{\mathbf{C}}$ so that its normal bundle is isomorphic to its tangent bundle, and the leaves of $\mathcal{F}_2$ correspond to the tangent planes of Π, up to isotopy.

As a consequence of these constructions we get that each $\mathrm{SO}(n+1, \mathbf{R})$-orbit in $P^n_{\mathbf{C}}$ is at constant distance from both Q and Π. That is, they are the level sets of the functions "distance to Q" and "distance to Π". The squares of these functions are Bott-Morse functions on $P^n_{\mathbf{C}}$, whose critical set is $Q \cup \Pi$, a result in the spirit of [DR].

In Section 3 we look at the (now classical) theorem saying that $P^2_{\mathbf{C}}$ modulo conjugation is the sphere S^4. This theorem has a long and remarkable history. As explained by V.I. Arnold in [Ar4], he was informed by Rokhlin that this result was known to Pontryagin in the 1930s. The first time this result appeared in print was in 1971, in [Ar1; p. 175], where Arnold used it to study real algebraic curves in $P^2_{\mathbf{R}}$. It was explained to us by Professor Arnold that at the time, it appeared to him that this was an obvious fact which had to be well known, so he stated it without proof. To his surprise, he found no mention of this theorem in the literature, so he asked a number of experts whether they knew about it. In 1973–74 there appeared two independent proofs of this theorem $S^4 \cong P^2_{\mathbf{C}}/conjugation$, given by W. Massey and N. Kuiper [Ku, Ma1]. So we call it the Arnold-Kuiper-Massey theorem (sometimes called the Kuiper-Massey theorem in the literature). Several other proofs of this result have been given by various authors afterwards, including important improvements and generalizations (see for instance [Mar, Mo, Va1, Va2]). We refer particularly to [Ar2], where Arnold gives his original proof, providing a real algebraic map $P^2_{\mathbf{C}} \to S^4$ that induces a diffeomorphism $P^2_{\mathbf{C}}/conjugation \cong S^4$, and to [Ar3, Ar4], where he gives several interesting generalizations following the same method. Different proofs, also with very remarkable generalizations, were given recently by M.F. Atiyah and E. Witten [AW], and by[1]) M.F. Atiyah and J. Berndt [AB].

Here we prove an equivariant version of this theorem, showing that the equivalence $P^2_{\mathbf{C}}/j \cong S^4$ can be realized by a real algebraic map Φ which conjugates the natural cohomogeneity 1 actions of $\mathrm{SO}(3,\mathbf{R})$ on $P^2_{\mathbf{C}}$ and S^4. Our proof is quite elementary: it uses only linear algebra. The key point is to give appropriate interpretations of $P^2_{\mathbf{C}}$ and S^4. In the case of the sphere, this is given in [HL, DR], where it is observed that S^4 is the set of matrices with norm 1 in the space $\mathcal{S} \cong \mathbf{R}^5$ of real symmetric (3×3) matrices with trace 0. Similarly, $P^2_{\mathbf{C}}$ is the space of complex Hermitian symmetric (3×3) matrices with trace 1 and satisfying $H^2 = H$, i.e. they are orthogonal projections into complex lines (a fact which is well known to the physicists since these lines correspond to states in quantum physics). The map Φ is the one that carries a matrix $H \in P^2_{\mathbf{C}}$ into the unit vector $\psi(H)/||\psi(H)|| \in \mathcal{S}$, where $\psi(H)$ is $[\frac{1}{3}I - \Re(H)]$, I is the (3×3) identity matrix and $\Re$ denotes the real part.

Finally, in Section 4 we use the results and constructions of the previous sections to construct interesting isometric orthogonal actions on $P^3_{\mathbf{C}}$ and S^7, as well as interesting Bott-Morse functions on such manifolds. For this we use the twistor fibration $P^3_{\mathbf{C}} \to S^4$ of Calabi-Penrose, that we describe below, and the beautiful geometry of the quaternions. We also describe the complement of $P^2_{\mathbf{R}}$ in $P^4_{\mathbf{R}}$, embedded as the image of the classical Veronese embedding $P^2_{\mathbf{R}} \hookrightarrow S^4$, followed by the canonical projection of S^4 onto $P^4_{\mathbf{R}}$.

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[1]) We thank Professor Atiyah for explaining to us that our proof is essentially the same as the one in [AB]. This extends to the quaternionic and Cayley planes, which provides corresponding theorems.

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1. On the topology of a quadric in $P^n_{\mathbf{C}}$

Let Q be a codimension 1, non-singular complex quadric in the projective space $P^n_{\mathbf{C}}$.

THEOREM 1.1. *The complement of Q in $P^n_{\mathbf{C}}$ is diffeomorphic to the total space of the tangent bundle of the n-dimensional real projective space:*

$$P^n_{\mathbf{C}} \setminus Q \cong T(P^n_{\mathbf{R}}) .$$

Proof. We first notice that a non-singular hypersurface of degree d in $P^n_{\mathbf{C}}$ is determined by a homogeneous polynomial of degree d in $n+1$ complex variables, with no critical points outside $0 \in \mathbf{C}^{n+1}$. Let $\mathcal{P}$ be the projective space of coefficients of homogeneous polynomials of degree d in $n+1$ complex variables. The general homogeneous equation of degree d in $n+1$ variables is

$$\sum_{\alpha_0+\cdots+\alpha_n=d} a_{\alpha_0,\ldots,\alpha_n} z_0^{\alpha_0} \ldots z_n^{\alpha_n} = 0 .$$

This defines a polynomial, and hence a hypersurface $\mathcal{X}$, in $\mathcal{P} \times P^n_{\mathbf{C}}$. The family of projective hypersurfaces of degree d in $P^n_{\mathbf{C}}$ is given by the map

$$\mathcal{E} : \mathcal{X} \to \mathcal{P} ,$$

induced by the projection of $\mathcal{P} \times P^n_{\mathbf{C}}$ onto $\mathcal{P}$. In $\mathcal{P}$, the polynomials defining singular hypersurfaces in $P^n_{\mathbf{C}}$ form a closed subvariety of complex codimension one. Hence its complement Ω is connected. Since the map $\mathcal{E}$ is a locally trivial fibration over Ω, by Ehresmann's lemma, one knows that any non-singular hypersurface $X \subset P^n_{\mathbf{C}}$ of degree d is ambient isotopic to the hypersurface defined by the Fermat polynomial $\mathbf{F}^n_d := z_0^d + \cdots + z_n^d$. That is, up to isotopy we can assume that X is the projectivization of the affine variety $V := \left\{z_0^d + \cdots + z_n^d = 0\right\}$ after removing the singular point $0 \in V$ (cf. [LC; Lemme 2.2]).

The projective space $P^n_{\mathbf{C}}$ is obtained dividing $\mathbf{C}^{n+1} - \{0\}$ by the $\mathbf{C}^*$-action:

$$g_t(z_0, \ldots, z_n) = (e^{it} z_0, \ldots, e^{it} z_n) , \qquad t \in \mathbf{C}^* = \mathbf{C} \setminus \{0\} .$$