1. Introduction

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PROJECTIVE GEOMETRY OF POLYGONS AND DISCRETE 4-VERTEX AND 6-VERTEX THEOREMS

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ABSTRACT. This paper is concerned with discrete versions of three well-known results from projective differential geometry: the four-vertex theorem, the theorem on six affine vertices, and Ghys' theorem on four zeroes of the Schwarzian derivative. We study the geometry of closed polygonal lines in $\mathbf{RP}^d$ and prove that polygons satisfying a certain convexity condition have at least $d+1$ flattenings. This result provides a new approach to the classical theorems mentioned.

1. INTRODUCTION

A *vertex* of a smooth plane curve is a point where the curve has 4^{th}-order contact with a circle (at a generic point the osculating circle has 3^{rd}-order contact with the curve). An *affine vertex* (or *sextactic point*) of a smooth plane curve is a point of 6^{th}-order contact with a conic. In 1909 S. Mukhopadhyaya [10] published two celebrated theorems, which are joined in the following statement:

Every closed smooth convex plane curve has at least 4 *distinct vertices and at least* 6 *distinct affine vertices.*

These results generated an extensive literature. From a modern point of view they are related, among other subjects, to the global singularity theory of wave fronts and to Sturm theory – see e.g. [1, 2, 4, 8, 17, 18] and references therein.

A recent and unexpected result along these lines is the following theorem due to E. Ghys [7]:

The Schwarzian derivative of every diffeomorphism of the projective line has at least 4 distinct zeroes.

(see also [11, 15, 6]). The Schwarzian derivative vanishes when the 3^{rd} jet of the diffeomorphism coincides with that of a projective transformation (at a generic point a diffeomorphism can be approximated by a projective transformation up to the 2^{nd} derivative). Ghys' theorem can be interpreted as the 4-vertex theorem in Lorentzian geometry (cf. references above).

The goal of this note is to study polygonal analogues of the above results. In our opinion, such a discretization of smooth formulations is interesting for the following reasons. Firstly, a discrete theorem is a priori stronger; it becomes, in the limit, a smooth one, thus providing a new proof of the latter. An important feature of the discrete approach is the availability of mathematical induction, which can considerably simplify the proofs. Secondly, the very operation of discretization is non-trivial: a single smooth theorem may lead to non-equivalent discrete ones. An example of this phenomenon is provided by two recent versions of the 4-vertex theorem for convex plane polygons [12, 13, 19, 16] – see Remark 2.4 below. To the best of our knowledge, these results are the only available discrete versions of the 4-vertex theorem.

In this regard we would like to draw attention to a famous lemma of Cauchy (1813):

Given two convex (plane or spherical) polygons whose respective sides are congruent, the cyclic sequence of the differences of respective angles of the polygons changes sign at least 4 times.

This result plays a crucial role in the proof of rigidity for convex polyhedra (see [5] for a survey). The Cauchy lemma implies, in the limit, the smooth 4-vertex theorem and can be viewed as the first result in the area under discussion.