

## 2. Symplectic characteristic classes

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## 2. SYMPLECTIC CHARACTERISTIC CLASSES

## 2.1 CHARACTERISTIC CLASSES AND REPRESENTATIONS

The previously defined homomorphism  $\phi: U(n) \rightarrow \mathrm{Sp}(2n, \mathbf{R})$  induces

$$H^*(B \mathrm{Sp}(2n, \mathbf{R}), \mathbf{Z}) \xrightarrow{\cong} H^*(B U(n), \mathbf{Z})$$

such that for  $j = 1, \dots, n$  the symplectic class  $d_j \in H^{2j}(B \mathrm{Sp}(2n, \mathbf{R}), \mathbf{Z})$  maps (per definition) to the universal Chern class  $c_j \in H^{2j}(B U(n), \mathbf{Z})$ . It is well-known that

$$H^*(B U(n), \mathbf{Z}) = \mathbf{Z}[c_1, \dots, c_n],$$

$$H^*(B \mathrm{Sp}(2n, \mathbf{R}), \mathbf{Z}) = \mathbf{Z}[d_1, \dots, d_n].$$

The class  $d_j = d_j(\mathbf{R})$  restricts to  $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z})$  for  $j = 1, \dots, n$ . Note that, strictly speaking, the class  $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z})$  depends also on  $n$ . But Charney has proven in [7] that for  $n > 2j + 4$  the inclusion

$$\mathrm{Sp}(2n, \mathbf{Z}) \longrightarrow \mathrm{Sp}(2n + 2, \mathbf{Z})$$

induces an isomorphism

$$H_j(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z}) \xrightarrow{\cong} H_j(\mathrm{Sp}(2n + 2, \mathbf{Z}), \mathbf{Z}).$$

It is a consequence of the universal coefficient theorem that her result implies the existence of an isomorphism

$$H^j(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z}) \xrightarrow{\cong} H^j(\mathrm{Sp}(2n + 2, \mathbf{Z}), \mathbf{Z})$$

for  $n > 2j + 4$ . This implies that  $H^{2j}(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z})$  is independent of  $n$  for  $n > 4j + 4$ . Representations

$$\rho: \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathrm{Sp}(2n, \mathbf{Z}),$$

$$\tilde{\rho}: \mathbf{Z}/p\mathbf{Z} \longrightarrow U(n)$$

induce homomorphisms

$$\rho^*: H^*(\mathrm{Sp}(2n, \mathbf{Z}), \mathbf{Z}) \longrightarrow H^*(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}),$$

$$\tilde{\rho}^*: H^*(B U(n), \mathbf{Z}) \longrightarrow H^*(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}).$$

We define  $d_j(\rho) := \rho^* d_j(\mathbf{Z})$ , the symplectic class of the representation  $\rho$ , and  $c_j(\tilde{\rho}) := \tilde{\rho}^*(c_j)$ , the Chern class of the representation  $\tilde{\rho}$ . We can consider any representation  $\tilde{\rho}$  of  $\mathbf{Z}/p\mathbf{Z}$  in  $U(n)$  as a representation  $\phi \circ \tilde{\rho}$  of  $\mathbf{Z}/p\mathbf{Z}$  in  $\mathrm{Sp}(2n, \mathbf{R})$ . We say that  $\tilde{\rho}$  factors through  $\mathrm{Sp}(2n, \mathbf{Z})$  if the image

$\phi(\tilde{\rho}(z))$  of any generator  $z \in \mathbf{Z}/p\mathbf{Z}$  is conjugate to a  $Y \in \mathrm{Sp}(2n, \mathbf{Z})$ . Then  $d_j(\rho) = \tilde{\rho}^*(c_j) = c_j(\rho)$ . We define the total Chern class of a representation  $\tilde{\rho}$  to be

$$c(\tilde{\rho}) := 1 + c_1(\tilde{\rho}) + c_2(\tilde{\rho}) + \cdots + c_n(\tilde{\rho}).$$

It has the well-known properties  $c(\rho \oplus \sigma) = c(\rho)c(\sigma)$ ,  $c(m\rho) = c(\rho)^m$ , where  $\rho, \sigma$  are representations and  $m$  is a positive integer.

## 2.2 SYMPLECTIC CHARACTERISTIC CLASSES AND CHERN CLASSES

**THEOREM 2.1.** *Let  $p$  be an odd prime. Then for any  $n = 1, \dots, (p-1)/2$  there exists a representation  $\tilde{\rho}: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{U}((p-1)/2)$  such that the  $n$ -th Chern class  $c_n(\tilde{\rho})$  is nonzero and the representation  $\phi \circ \tilde{\rho}: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{R})$  factors, up to conjugation, through a representation  $\rho: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{Z})$ .*

The representation  $\tilde{\rho}$  factors through  $\mathrm{Sp}(p-1, \mathbf{Z})$  if the image  $\tilde{\rho}(z)$  of a generator  $z \in \mathbf{Z}/p\mathbf{Z}$  satisfies the condition stated in Theorem 1.2. Then, because  $c_n(\tilde{\rho}) \neq 0$ , we have  $d_n(\rho) \neq 0$  where  $\rho: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{Z})$  is the representation corresponding to  $\tilde{\rho}$ .

*Proof of Theorem 2.1.* Let  $\mathcal{U}$  be the set of subsets  $\mathcal{I} \subset (\mathbf{Z}/p\mathbf{Z})^*$  of cardinality  $|\mathcal{I}| = (p-1)/2$ , and  $j \in \mathcal{I}$  implies  $p-j \notin \mathcal{I}$ . The cardinality of  $\mathcal{U}$  is  $2^{(p-1)/2}$ . We always assume the elements  $j \in \mathcal{I}$  to be represented by integers  $j$  with  $1 \leq j < p$ . Note that we will use the same notation for the elements of  $\mathcal{I}$  and their representatives. For  $j = 1, \dots, p-1$  let  $\tilde{\rho}_j: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{U}(1)$  be the one-dimensional representation with  $\tilde{\rho}_j(z) := e^{j2\pi i/p}$  for a fixed generator  $z \in \mathbf{Z}/p\mathbf{Z}$ . For a given  $\mathcal{I}$  we define  $\tilde{\rho}_{\mathcal{I}}$  to be the direct sum of the representations  $\tilde{\rho}_j$ ,  $j \in \mathcal{I}$ . Let  $x := c_1(\tilde{\rho}_1)$ , then the total Chern class of  $\tilde{\rho}_{\mathcal{I}}$  is

$$c(\tilde{\rho}_{\mathcal{I}}) = c\left(\bigoplus_{j \in \mathcal{I}} \tilde{\rho}_j\right) = \prod_{j \in \mathcal{I}} (1 + jx).$$

The representations  $\tilde{\rho}_{\mathcal{I}}$  are those which factor through  $\mathrm{Sp}(p-1, \mathbf{Z})$ . For a given  $\mathcal{I} \in \mathcal{U}$  we define  $-\mathcal{I} := \{p-j \mid j \in \mathcal{I}\}$ . Then  $-\mathcal{I} \in \mathcal{U}$  and  $\mathcal{I} \cup -\mathcal{I} = (\mathbf{Z}/p\mathbf{Z})^*$ . Moreover, we get  $c(\tilde{\rho}_{\mathcal{I}})c(\tilde{\rho}_{-\mathcal{I}}) = 1 - x^{p-1}$ . The  $n$ -th Chern class  $c_n(\tilde{\rho}_{\mathcal{I}})$  is nonzero if and only if the coefficient  $a_n$  of  $x^n$  in the total Chern class  $c(\tilde{\rho}_{\mathcal{I}})$  is nonzero. Let  $\mathcal{I} := \{j_1, \dots, j_{(p-1)/2}\} \in \mathcal{U}$ ; then we define

$$\mathcal{I}_l := \{j_1, \dots, j_{l-1}, -j_l, j_{l+1}, \dots, j_{(p-1)/2}\} \in \mathcal{U}.$$

We assume that  $1 \leq n \leq (p-1)/2$  exists such that for each set  $\mathcal{I} \in \mathcal{U}$  the coefficient  $a_n$  of  $x^n$  in  $c(\tilde{\rho}_{\mathcal{I}})$  is zero. It is impossible that  $n = (p-1)/2$

because  $a_{(p-1)/2}$  is the product of the  $j \in \mathcal{I}$  and therefore nonzero. Now let  $n \neq 0$ ,  $n \neq (p-1)/2$ ; then we define for any  $l = 1, \dots, (p-1)/2$

$$b_n^l := \sum_{\substack{J \subseteq \mathcal{I} \setminus \{j_l\} \\ |J|=n}} \prod_{j \in J} j, \quad b_0^l := 1.$$

Then the coefficient of  $x^n$  in  $c(\tilde{\rho}_{\mathcal{I}})$  is  $a_n = b_n^l + j_l b_{n-1}^l$ . Because of our assumption, the coefficients of  $x^n$  in  $c(\tilde{\rho}_{\mathcal{I}})$  and in  $c(\tilde{\rho}_{\mathcal{I}_l})$  are  $b_n^l + j_l b_{n-1}^l = 0$  and  $b_n^l - j_l b_{n-1}^l = 0$  respectively. This implies that  $b_n^l = 0$ ,  $b_{n-1}^l = 0$  and

$$\begin{aligned} a_{n+1} &= \sum_{\substack{J \subseteq \mathcal{I} \\ |J|=n+1}} \prod_{j \in J} j \\ &= \frac{1}{n+1} \sum_{j_l \in \mathcal{I}} \left( j_l \sum_{\substack{J \subseteq \mathcal{I} \setminus \{j_l\} \\ |J|=n}} \prod_{j \in J} j \right) = \frac{1}{n+1} \sum_{j_l \in \mathcal{I}} j_l b_n^l = 0. \end{aligned}$$

The factor  $1/(n+1)$  appears because in the second line we have  $n+1$  times each term appearing in the sum of the first line. Therefore  $a_{n+1} = 0$  for each set  $\mathcal{I} \in \mathcal{U}$ , and by induction we get  $a_{(p-1)/2} = 0$  for each set  $\mathcal{I} \in \mathcal{U}$ , which is impossible.  $\square$

Let  $\mathrm{Sp}(\mathbf{Z}) := \bigcup_{n \geq 1} \mathrm{Sp}(2n, \mathbf{Z})$ .

**THEOREM 2.2.** *For every  $j \geq 1$ ,  $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$  has infinite order.*

*Proof.* This theorem is a corollary of Theorem 2.1. A consequence of the stability result stated in section 2.1 is that for  $p-1 > 8j+8$  the inclusion

$$\mathrm{Sp}(p-1, \mathbf{Z}) \longrightarrow \mathrm{Sp}(\mathbf{Z})$$

induces an isomorphism

$$H^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z}) \xrightarrow{\cong} H^{2j}(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z}).$$

In Theorem 2.1 we have shown that for any odd prime  $p$  and any integer  $j = 1, \dots, (p-1)/2$  a representation  $\tilde{\rho}_{\mathcal{I}}: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{U}((p-1)/2)$  exists that factors through  $\mathrm{Sp}(p-1, \mathbf{Z})$  and for which the  $j$ -th Chern class  $c_j(\tilde{\rho}_{\mathcal{I}})$  is nonzero. Then the  $j$ -th symplectic class  $d_j(\rho_{\mathcal{I}})$  is also nonzero. Here the

representation  $\rho_I: \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{Sp}(p-1, \mathbf{Z})$  is the one corresponding to  $\tilde{\rho}_I$ . We have an induced homomorphism

$$\begin{aligned} \rho_I^*: H^{2j}(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z}) &\longrightarrow H^{2j}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}) \\ d_j(\mathbf{Z}) &\longmapsto d_j(\rho_I). \end{aligned}$$

Herewith for any  $p$  the class  $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(p-1, \mathbf{Z}), \mathbf{Z})$  is nonzero and has either infinite order or finite order divisible by  $p$ , since it restricts non-trivially to  $H^{2j}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z})$ . This shows that  $d_j(\mathbf{Z}) \in H^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$  has infinite order.  $\square$

This is a new proof of a result of A. Borel [3]. He proved that  $H^*(\mathrm{Sp}(\mathbf{Z}), \mathbf{Q}) = \mathbf{Q}[d_1, d_3, \dots]$ . Moreover, each  $d_{2i}$  can be expressed as a polynomial in the  $d_{2j+1}$ 's. This implies that all the  $d_i(\mathbf{Z})$ 's have infinite order.

#### REFERENCES

- [1] ALEXANDER, J. P., P. E. CONNER and G. C. HAMRICK. *Odd order group actions and Witt classifications of innerproducts*. Lecture Notes in Mathematics 625. Springer, 1977.
- [2] ALEXANDER, J. P., P. E. CONNER, G. C. HAMRICK and J. W. VICK. Witt classes of integral representations of an abelian  $p$ -group. *Bull. Amer. Math. Soc.* 80 (1974), 1179–1182.
- [3] BOREL, A. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)* 7 (1974), 235–272.
- [4] BROWN, K. S. Euler characteristics of discrete groups and  $G$ -spaces. *Invent. Math.* 27 (1974), 229–264.
- [5] BÜRGISSER, B. Elements of finite order in symplectic groups. *Arch. Math. (Basel)* 39 (1982), 501–509.
- [6] BUSCH, C. Symplectic characteristic classes and the Farrell cohomology of  $\mathrm{Sp}(p-1, \mathbf{Z})$ . Dissertation ETH no. 13506, ETH Zurich (2000).
- [7] CHARNEY, R. A generalization of a theorem of Vogtmann. *J. Pure Appl. Algebra* 44 (1987), 107–125.
- [8] GARBANATI, D. A. Unit signatures, and even class numbers, and relative class numbers. *J. reine angew. Math.* 274–275 (1975), 376–384.
- [9] NEUKIRCH, J. *Algebraische Zahlentheorie*. Springer, 1992.
- [10] ———. *Class Field Theory*. Grundlehren der mathematischen Wissenschaften 280. Springer, 1986.
- [11] SJERVE, D. and Q. YANG. Conjugacy classes of  $p$ -torsion in  $\mathrm{Sp}_{p-1}(\mathbf{Z})$ . *J. Algebra* 195 (1997), 580–603.