# 1.1 DEFINITION

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# SYMPLECTIC CHARACTERISTIC CLASSES

by Cornelia BUSCH

ABSTRACT. We present a new proof of the fact that the universal symplectic classes $d_j(\mathbf{Z}) \in \mathrm{H}^{2j}(\mathrm{Sp}(\mathbf{Z}), \mathbf{Z})$ have infinite order. This proof uses only techniques from group cohomology. In order to obtain this result, we determine representations $\mathbf{Z}/p\mathbf{Z} \to \mathrm{U}((p-1)/2)$ whose associated representation $\mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{R})$ factors, up to conjugation, through a representation $\mathbf{Z}/p\mathbf{Z} \to \mathrm{Sp}(p-1, \mathbf{Z})$.

In this article we prerequire some basic notions from the theory of cyclotomic fields. For the reader who is not familiar with this subject we recommend the books of Washington [12] and of Neukirch [9]. An introduction to the arithmetical part is also given in my thesis [6].

This article presents a result of my doctoral thesis, which I wrote at the ETH Zurich under the supervision of G. Mislin, whom I want to thank for his excellent support.

## 1. THE SYMPLECTIC GROUP

### 1.1 DEFINITION

Let $R$ be a commutative ring with $1$. The general linear group $\mathrm{GL}(n, R)$ is defined to be the multiplicative group of invertible $n \times n$-matrices over $R$.

DEFINITION. The *symplectic group* $\mathrm{Sp}(2n, R)$ over the ring $R$ is the subgroup of matrices $Y \in \mathrm{GL}(2n, R)$ that satisfy

$$Y^{\mathrm{T}} J Y = J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where $I_n$ is the $n \times n$-identity matrix.

It is the group of isometries of the skew-symmetric bilinear form

$$\begin{aligned} \langle\ ,\ \rangle \colon R^{2n} \times R^{2n} &\longrightarrow R \\ (x,y) &\longmapsto \langle x,y\rangle := x^{\mathrm{T}} J y\,. \end{aligned}$$

It follows from a result of Bürgisser [5] that elements of odd prime order $p$ exist in $\mathrm{Sp}\,(2n, \mathbf{Z})$ if and only if $2n \geqslant p-1$.

PROPOSITION 1.1. *The eigenvalues of a matrix $Y \in \mathrm{Sp}\,(p-1, \mathbf{Z})$ of odd prime order $p$ are the primitive $p$-th roots of unity, hence the zeros of the polynomial*

$$m(x) = x^{p-1} + \cdots + x + 1\,.$$

*Proof.* If $\lambda$ is an eigenvalue of $Y$, we have $\lambda = 1$ or $\lambda = \xi$, a primitive $p$-th root of unity, and the characteristic polynomial of $Y$ divides $x^p - 1$ and has integer coefficients. Since $m(x)$ is irreducible over $\mathbf{Q}$, the claim follows. $\square$

### 1.2 A RELATION BETWEEN $\mathrm{U}\left(\frac{p-1}{2}\right)$ AND $\mathrm{Sp}\,(p-1, \mathbf{Z})$

Let $X \in \mathrm{U}(n)$, i.e., $X \in \mathrm{GL}(n, \mathbf{C})$ and $X^*X = I_n$ where $X^* = \overline{X}^{\mathrm{T}}$ and $I_n$ is the $n \times n$-identity matrix. We can write $X = A + iB$ with $A, B \in \mathrm{M}(n, \mathbf{R})$, the ring of real matrices. We now define the following map

$$\begin{aligned} \phi\colon\ \mathrm{U}(n) &\longrightarrow \mathrm{Sp}\,(2n, \mathbf{R}) \\ X = A + iB &\longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} =: \phi(X)\,. \end{aligned}$$

The map $\phi$ is an injective homomorphism. Moreover, it is well-known that $\phi$ maps $\mathrm{U}(n)$ onto a maximal compact subgroup of $\mathrm{Sp}\,(2n, \mathbf{R})$. In this section we will prove the following theorem.

THEOREM 1.2. *Let $X \in \mathrm{U}((p-1)/2)$ be of odd prime order $p$. We define $\phi\colon \mathrm{U}((p-1)/2) \to \mathrm{Sp}\,(p-1, \mathbf{R})$ as above. Then $\phi(X) \in \mathrm{Sp}\,(p-1, \mathbf{R})$ is conjugate to $Y \in \mathrm{Sp}\,(p-1, \mathbf{Z})$ if and only if the eigenvalues $\lambda_1, \ldots, \lambda_{(p-1)/2}$ of $X$ are such that*

$$\left\{\lambda_1, \ldots, \lambda_{(p-1)/2}, \overline{\lambda}_1, \ldots, \overline{\lambda}_{(p-1)/2}\right\}$$

*is a complete set of primitive $p$-th roots of unity.*