5. Proof of the theorem

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of S. In particular, the cardinality of the quotient of the isometry group of S under the subgroup fixing a given systole equals $6g+3$.

To complete the proof of our proposition we have to investigate the ideal surfaces S_∞ associated to simple triangle surfaces $S(p;k)$. The above considerations are equally valid for these surfaces and show that S_∞ has more than $4g+4$ systoles if and only if p divides $k(k-1)+1$ and if the length ℓ_0 of a lift of a side pairing orbit for S_∞ is not bigger than $6 \operatorname{arccosh} \frac{3}{2}$. An explicit computation shows as before that this is the case if and only if S_∞ is associated to one of the surfaces $S(7;3), S(13;4), S(21;5)$. $\square$

5. Proof of the theorem

Using the notation of Lemma 2.2, our goal is to show that the triangle surfaces $S(7;3), S(13;4), S(21;5)$ and their associated ideal surfaces are maximal. Following Schmutz [S1], for this it is enough to show that for each of these surfaces S the Teichmüller space is parametrized in a neighborhood of S by the lengths of those closed geodesics which are freely homotopic to a systole on S.

Let for the moment $p \geq 5$ be an arbitrary odd number and let $k \in \{2, \ldots, p-1\}$ be such that k and $k-1$ are prime to p. Write $g = (p-1)/2$. As in the introduction let $\mathcal{T}_{g,3}$ be the Teichmüller space of surfaces of genus g with 3 punctures. Let $S = S(p;k)$ and let S_∞ be the ideal surface associated to S. The basic group Γ of orientation preserving isometries of S acts as a group of isometries on the surface S_∞.

It will be useful to give a geometric description of S_∞. For this let $\triangle_\infty$ be an ideal triangle in $\mathbf{H}^2$ and let $T \subset \triangle_\infty$ be the finite equilateral triangle inscribed in $\triangle_\infty$ which is invariant under all isometries of $\triangle_\infty$. The vertices of T determine a distinguished point on each side of $\triangle_\infty$.

There is a unique way to glue $2p$ copies of $\triangle_\infty$ to a disc A with one puncture in its interior and $2p$ punctures on the boundary in such a way that the glueing maps identify the distinguished points on the sides of $\triangle_\infty$. The boundary of A then consists of $2p$ geodesic lines. Each of the triangles which makes up A contains exactly one of these boundary geodesics. We number the boundary geodesics in counter clockwise order and glue the $2i+1$-th geodesic to the $2i+2k$-th geodesic by an orientation reversing isometry which identifies the distinguished points on these geodesics. The resulting surface is the ideal surface S_∞ associated to S. Notice that S_∞ admits a canonical triangulation into ideal triangles which corresponds to the canonical triangulation of S.

Denote by $0, 1, 2$ the cusps of S_∞. There are p edges of the canonical triangulation which connect the cusp 0 to the cusp 1. There is a natural counter clockwise ordering of these edges which is induced by the ordering of the ideal triangles around the cusp 0. We denote by α_i^0 the i-th edge with respect to this ordering and orient it in such a way that it goes from 0 to 1. Similarly we define α_i^1 to be the i-th edge of our triangulation with respect to the counter-clockwise ordering around the cusp 1 which goes from the cusp 1 to the cusp 2. Let also α_i^2 be the i-th edge ordered around the cusp 2 which goes from the cusp 2 to the cusp 0.

Each marked surface of genus $g = (p-1)/2$ with three punctures can be triangulated by $2p$ ideal triangles. If we cut the surface open along the edges of this triangulation, then we obtain $2p$ ideal triangles. To get the surface back we glue the triangles along their boundary geodesics in the fixed combinatorial pattern as above. The different points in $\mathcal{T}_{g,3}$ then differ by the way this glueing is arranged.

Namely, for each glueing we have one degree of freedom which corresponds to a left earthquake path along one of the geodesic arcs α_i^j. Using the marking given by the distinguished points on the boundary of an ideal triangle and the induced boundary orientation, the glueings of an ordered pair (T_1, T_2) of (oriented) ideal triangles along a boundary geodesic can be parametrized by a real (left) sliding parameter. The glueing which identifies the distinguished points corresponds to the parameter 0. A glueing where the distinguished point on the boundary geodesic of the triangle T_1 is mapped to the right of the distinguished point on the boundary geodesic of the triangle T_2 corresponds to a positive sliding parameter.

Following Thurston [T], in order to obtain a complete hyperbolic surface from the $3p$ glueings of $2p$ ideal triangles in the above combinatorial way, it is necessary and sufficient that at each of the three punctures of the resulting surface the sum of all the sliding parameters for all geodesics which go to this puncture vanishes. Thus if we denote by $V \subset \mathbf{R}^p$ the linear subspace of all vectors which are orthogonal to the vector $(1, \dots, 1)$, then there is a natural bijection of $\mathcal{T}_{g,3}$ onto $V^3 = V \times V \times V$ which maps a surface $M \in \mathcal{T}_{g,3}$ to its $3p$-tuple of sliding parameters.

Let now γ_i^0 be the piecewise geodesic in S_∞ which consists of the arc α_i^0 with the orientation reversed and the arc α_{i+k}^0. If we compactify the surface S_∞ by adding a point at each puncture, then the compactification of γ_i^0 is a simple closed curve on $S = S(p; k)$ which is freely homotopic to the closed geodesic ψ_i^0 on S obtained by projecting a geodesic in a fundamental $2p$-gon Ω which connects the midpoints of the edges $2i+1$ and $2i+2k$. Similarly, let $k(1), k(2) \in \{2, \dots, p-1\}$ be such that $k(1)(k-1)+1 \equiv 0 \mod p$ and $k(k(2)-1)+1 \equiv 0 \mod p$ and denote for $j = 1, 2$ by γ_i^j the piecewise geodesic which consists of the geodesic α_i^j with the reversed orientation and the geodesic $\alpha_{i+k(j)}^j$. Write also $k(0) = k$.

An earthquake path through S_∞ induced by the curve γ_i^j deforms the surface S_∞ by a family of glueings with sliding parameter $-t$ along α_i^j, sliding parameter t along $\alpha_{i+k(j)}^j$ ($t \in \mathbf{R}$) and sliding parameter 0 otherwise and hence this earthquake path gives rise to a smooth (in fact real analytic) curve in $\mathcal{T}_{g,3}$. From this observation the following is immediate.

LEMMA 5.1. *For every surface $M \in \mathcal{T}_{g,3}$ the tangents of the earthquake paths along the curves γ_i^j span the tangent space of $\mathcal{T}_{g,3}$ at M.*

Proof. Let $M \in \mathcal{T}_{g,3}$ and denote by ξ_i^j the tangent at M of the earthquake path along α_i^j. We observed above that there is a linear isomorphism of the vector space V^3 onto the tangent space of $\mathcal{T}_{g,3}$ at M which maps the point $(0_1, \dots, 0_p, a_1, \dots, a_p, b_1, \dots, b_p) \in V^3$ to the tangent vector $\sum_{i,j} j_i \xi_i^j$. Since the tangent at M of the earthquake path induced by γ_i^j is just $\xi_{i+k(j)}^j - \xi_i^j$ the lemma follows. $\square$

There is a natural real analytic submersion P of $\mathcal{T}_{g,3}$ onto $\mathcal{T}_g$ which is equivariant under the action of the basic group Γ. This submersion simply maps a surface of genus g with 3 punctures to the surface obtained by compactifying each puncture with a single point. For every $S \in \mathcal{T}_g$ the fibre of P over S consists of all surfaces in $\mathcal{T}_{g,3}$ which we obtain from S by removing an ordered triple of pairwise distinct points. In particular, the fibre is a real analytic submanifold of $\mathcal{T}_{g,3}$ of dimension 6. We denote by W the 6-dimensional subbundle of the tangent bundle of $\mathcal{T}_{g,3}$ which is the kernel of the differential of P. This bundle has a natural direct decomposition $W = W_0 \oplus W_1 \oplus W_2$ into two-dimensional subbundles W_j. Here the bundle W_j is the tangent bundle of the fibres of the fibration $\mathcal{T}_{g,3} \to \mathcal{T}_{g,2}$ which we obtain by adding for every surface $M \in \mathcal{T}_{g,3}$ a single point at the punture j of M.

For $M \in \mathcal{T}_{g,3}$ the compactifications of the curves $P\gamma_i^j$ are homotopically nontrivial simple closed curves on PM. There is a unique free homotopy class on M which can be represented by a closed curve which does not intersect γ_i^j and whose projection to PM is freely homotopic to the compactification of $P\gamma_i^j$. We denote by $\widetilde{\psi}_i^j$ the unique geodesic on M representing this class. We have.

LEMMA 5.2. *Let ξ_i^j, ζ_i^j be the tangent of the earthquake path along $\widetilde{\psi}_i^j, \gamma_i^j$. Then there are functions $a_i^j : \mathcal{T}_{g,3} \to \mathbf{R}$ such that $\zeta_i^j - a_i^j \xi_i^j \in W_j \oplus W_{j+1}$.*

Proof. Let $M \in \mathcal{T}_{g,3}$ and for $i \in \{1, \ldots, p\}, j = 0, 1, 2$ consider the piecewise geodesic γ_i^j and the geodesic $\widetilde{\psi}_i^j$ on M. Since the number of intersections between γ_i^j and $\widetilde{\psi}_i^j$ is the minimum of the number of intersections between γ_i^j and any curve which is freely homotopic to $\widetilde{\psi}_i^j$, the geodesics $\widetilde{\psi}_i^j$ and γ_i^j on M do not intersect. If we cut the surface M open along the curves γ_i^j and $\widetilde{\psi}_i^j$ then the interior of one of the connected surfaces with boundary which we obtain in this way, say the surface C, is homeomorphic to an open annulus. One boundary component of C is the curve $\widetilde{\psi}_i^j$, the second boundary component has two punctures and consists of the curve γ_i^j.

By construction, the curve $\widetilde{\psi}_i^j$ is non-separating and therefore there is a simple closed geodesic η on M which neither intersects γ_i^j nor $\widetilde{\psi}_i^j$ and such that after cutting M along η we obtain two bordered surfaces S_1, S_2. The surface S_1 is a surface of genus 1 with one geodesic boundary circle and two punctures in its interior and contains the annulus C bounded by the curves γ_i^j and $\widetilde{\psi}_i^j$. The earthquake paths along the piecewise geodesic γ_i^j and the geodesic $\widetilde{\psi}_i^j$ leave the hyperbolic length of a closed geodesic σ on M fixed if and only if σ does not have a transverse intersection with $\gamma_i^j, \widetilde{\psi}_i^j$. Thus these earthquake paths define deformations of the hyperbolic structure on S_1 leaving the length of the boundary fixed.

The Teichmüller space of marked hyperbolic structures on the bordered torus S_1 with two punctures and a boundary geodesic of fixed length is 6-dimensional. Its tangent bundle contains a 5-dimensional subbundle V which consists of all infinitesimal deformations preserving the modulus of a maximal (twice punctured) ring domain with core curve homotopic to $\widetilde{\psi}_i^j$.

We claim that this 5-dimensional subbundle V contains the tangents of the earthquake paths along the geodesic $\widetilde{\psi}_i^j$ and along the piecewise geodesic γ_i^j.

To see this let ζ be the unique simple geodesic arc in S_1 which meets the boundary geodesic η perpendicularly and which neither intersects $\widetilde{\psi}_i^j$ nor γ_i^j. Let $\bar{S}_1$ be the compactification of S_1 which we obtain by simply adding one point at each puncture. If we cut $\bar{S}_1$ open along ζ, then we obtain a standard ring domain A normalized by the fixed choice of a height, say the height 1, with core curve homotopic to $\widetilde{\psi}_i^j$ and whose modulus is maximal among all ring domains with this property [St]. The boundary of A consists of two circles which contain each a copy of the arc ζ as well as a nontrivial component of the boundary geodesic η. We mark the arc on each boundary component which corresponds to the arc ζ. The surface $\bar{S}_1$ is obtained by glueing the two marked arcs on the two boundary components with the restriction of a complex linear map of the complex plane.

The compactification of γ_i^j is a closed curve in the ring domain A which is freely homotopic to the core curve. If we cut A open along this curve then by uniformization we obtain two standard ring domains A_1, A_2 with one common boundary circle. The earthquake path induced by γ_i^j consists in cutting A along the common boundary circle of A_1, A_2 and glueing the ring domains A_1, A_2 back with a new boundary identification. This procedure does not change the lengths of the arcs η or ζ nor the modulus of the annulus A. In other words, the tangent of this earthquake path is contained in V. The same argument applies to the earthquake path induced by the geodesic $\widetilde{\psi}_i^j$. We conclude that this earthquake path induces a nontrivial infinitesimal deformation of the conformal structure on the compactification of our bordered punctured torus which leaves the modulus of a maximal ring domain with core curve homotopic to $\widetilde{\psi}_i^j$ fixed. In particular, the tangent of this earthquake path is contained in V but not in the kernel of the differential of the natural map which assigns to a twice punctured bordered torus its compactification.

As a conclusion, the tangent at M of the earthquake path induced by γ_i^j can be written in the form $a_i^j \xi_i^j + \eta_i^j$ where ξ_i^j is the tangent of the earthquake path along $\widetilde{\psi}_i^j$, $a_i^j \in \mathbf{R}$ and η_i^j is contained in the bundle $W_j \oplus W_{j+1}$. This shows the lemma. $\square$

Let now $k \geq 3$ and consider again the ideal surface S_∞ associated to the simple triangle surface $S = S(k(k-1)+1; k)$. Using the above notation, for $m = jp + i$ $(j \in \{0, 1, 2\}, i < p)$ write $\widetilde{\psi}_m = \widetilde{\psi}_i^j$. For $M \in \mathcal{T}_{g,3}$ and $m \in \{1, \ldots, 3p\}$ denote by $\ell_M(\widetilde{\psi}_m)$ the length of the closed geodesic $\widetilde{\psi}_m$ on M. The functions $M \in \mathcal{T}_{g,3} \to \ell_M(\widetilde{\psi}_m)$ are real analytic [K]. This means that we obtain a real analytic map Ψ_∞ of $\mathcal{T}_{g,3}$ into $\mathbf{R}^{3p}$ by mapping a surface M to $\Psi_\infty(M) = (\ell_M(\widetilde{\psi}_1), \ldots, \ell_M(\widetilde{\psi}_{3p}))$. From Lemma 5.1 and Lemma 5.2 we conclude.

COROLLARY 5.3. *The map Ψ_∞ is of maximal rank differentiable at S_∞.*

Proof. Following Wolpert [W], the tangent of the earthquake path along $\widetilde{\psi}_i^j$ is dual with respect to the Weil Petersen Kähler form to the differential of the length function of $\widetilde{\psi}_i^j$ on $\mathcal{T}_{g,3}$. Thus to show the corollary it is enough to show that the tangent space of $\mathcal{T}_{g,3}$ at S_∞ is spanned by the tangents ξ_i^j of the earthquake paths along the curves $\widetilde{\psi}_i^j$.

Let G be the group of isometries of S_∞ which is generated by the basic group Γ and the group Σ of order 3 contained in the normalizer of Γ. The group G acts on the Teichmüller space $\mathcal{T}_{g,3}$ as a group of automorphism which fixes the surface S_∞.

Let Λ be the linear isometry of $\mathbf{R}^p$ defined in canonical coordinates by $\Lambda(x_1,\dots,x_p)=(x_2,\dots,x_p,x_1)$; then $\Lambda\times\Lambda\times\Lambda=\Lambda_3$ is a linear isometry of $\mathbf{R}^{3p}$. If J_1 is the canonical generator of the normal cyclic subgroup Γ of G then we have $\Psi_\infty(J_1M)=\Lambda_3\Psi_\infty(M)$.

Let τ be the linear isometry of $\mathbf{R}^{3p}=\mathbf{R}^p\times\mathbf{R}^p\times\mathbf{R}^p$ which cyclicly permutes the factors $\mathbf{R}^p$ in the direct decomposition of $\mathbf{R}^{3p}$. There is a permutation σ of $\{1,\dots,p\}$ of order $p-1$ with diagonal extension σ_3 to $\mathbf{R}^{3p}$ such that the canonical generator J_2 of the cyclic subgroup Σ acts by $\Psi_\infty J_2(M)=\sigma_3\circ\tau(\Psi_\infty M)$.

The eigenvalues of the linear isometry Λ are the p-th roots of unity. The eigenspace for the eigenvalue 1 is spanned by $(1,\dots,1)$ and the other generalized eigenspaces are of dimension 2. The map $\sigma_3\circ\tau$ permutes the generalized eigenspaces of the diagonal extension Λ_3 which correspond to eigenvalues different from 1 and acts as a cyclic group of permutations on the eigenspace Z of Λ^3 with respect to the eigenvalue 1. The orthogonal complement $Z^\perp$ of Z in $\mathbf{R}^{3p}$ decomposes into g irreducible invariant subspaces of dimension 6 each.

The surface S_∞ is a fixed point for the action of G. By Lemma 5.1, the tangent space of $\mathcal{T}_{g,3}$ at S_∞ as a G-space is isomorphic to $Z^\perp$, where the differential of J_1 acts as the map Λ_3 and the differential of J_2 as $\sigma_3\circ\tau$. The 6-dimensional tangent space W at S_∞ of the fibre of the fibration $P\colon\mathcal{T}_{g,3}\to\mathcal{T}_g$ is invariant under the action of G and for reasons of dimension necessarily irreducible.

Let as before ξ_i^j,ζ_i^j be the tangent at S_∞ of the earthquake path along $\widetilde{\psi}_i^j,\gamma_i^j$.

Denote by L the linear map which maps ζ_i^j to ξ_i^j. Then L is G-equivariant and by Lemma 5.2 its kernel is contained in the G-invariant space W. Since W is irreducible under G the kernel of L is either trivial or coincides with W.

We have to show that the latter does not hold. For this we have to find a tangent vector $X\in W$ such that $LX\neq 0$.

Consider the unit disc D in the complex plane with boundary circle S^1 and hyperbolic metric. Let D_∞ be the disc with the point 0 deleted. It carries a unique complete hyperbolic metric for which the puncture is a standard cusp. This metric admits an isometric circle action which induces the standard parametrization of the boundary circle $S^1=[0,2\pi)$.

Let Ω_0, Ω be the regular ideal hyperbolic $2p$-gon in D_∞, D whose set $\mathcal{P}$ of vertices consists of the points $j\pi/p$ $(j = 1, \ldots, 2p)$. These $2p$-gons admit a cyclic group of order $2p$ of isometries, and Ω_0 hence is isometric to the once punctured polygon which we obtain by cutting S_∞ along the geodesics of the canonical triangulation joining the cusps 1 and 2.

For an interior point x of $\widetilde{\Omega}$ consider the polygon $\Omega_x = \widetilde{\Omega} \setminus \{x\}$ with one puncture at x. The punctured polygon Ω_x carries a hyperbolic metric of finite volume such that the boundary consists of $2p$ geodesic lines, and it is naturally triangulated into $2p$ ideal triangles.

Let γ be a hyperbolic geodesic in D through $\gamma(0) = 0$. For every $t \in \mathbf{R}$ there is a unique hyperbolic isometry Ψ_t of D which fixes the endpoints of γ and maps $\gamma(t)$ to 0. The image under Ψ_t of the punctured polygon $\Omega_{\gamma(t)}$ is an ideal hyperbolic polygon with puncture at 0 and whose vertices on S^1 are the points in $\Psi_t\mathcal{P}$. The punctured polygon $\Psi_t\Omega_{\gamma(t)}$ can be obtained from Ω_0 by an earthquake deformation along the geodesics which joins 0 to the vertices of Ω_0 as follows.

Consider an ordered triple (a, b, c) of 3 pairwise distinct points on the boundary circle S^1 of D_∞ arranged in counter clockwise order. These points determine an ideal quadrangle Q which decomposes Q into 2 ideal hyperbolic triangles embedded in D_∞ which have one vertex at 0. Let T_1 be the triangle with vertices a, b, and let T_2 be the triangle with vertices b, c. If the euclidean distance between a and b is smaller than the distance between b and c then the glueing map which gives the quadrangle Q back from the triangles T_1 and T_2 maps the distinguished point of the boundary geodesic of T_1 to the right of the distinguished point on the boundary geodesic of T_2 with respect to the boundary orientation of T_2. In other words, with our above notation the glueing corresponds to a positive sliding parameter.

The derivative of the restriction of Ψ_t to S^1 has a maximum at its repelling fix point z_1 and a minimum at its attracting fix point z_2. It is strictly monotonous on each of the two components of $S^1 - \{z_1, z_2\}$. Let (z_1, z_2) be the component which corresponds to an open interval in $[0, 2\pi)$ with left endpoint z_1. The above analysis shows that the deformation of the polygon Ω_0 which defines $\Psi_t\Omega_{\gamma(t)}$ has a negative sliding parameter for every geodesic which joins 0 to a point in $\mathcal{P} \cap (z_1, z_2)$. The sliding parameter is positive for all geodesics which join 0 to a point in $\mathcal{P} \cap (z_2, z_1)$.

Choose now γ in such a way that its forward endpoint equals $k\pi/2p$ and that its backward endpoint equals $k\pi/2p + \pi$. Let ρ be the reflection of $\widetilde{\Omega}$ along γ. This reflection induces an orientation reversing isometry of D_∞ which commutes with the above deformation of Ω_0 along γ. Denote by β_i

the geodesic which connects the center 0 to $(k+i)\pi/2p$ $(1 \le i \le 2p)$ and let ν_i be the tangent of the earthquake path along β_i. By symmetry, the tangent at $t=0$ of our deformation of Ω_0 along γ can be written in the form $\sum a_i\nu_i$ where $a_i < 0$ and $a_{i-p} = -a_i$ for $i = 1, \dots, p-1$.

Consider now the geodesic $\widetilde{\psi}_1^0$. It intersects γ perpendicularly and has $2k-2 \ge 2$ additional intersections with the geodesics β_i. For $i \in \{1, \dots, k-1\}$ denote by δ_i the oriented angle of the intersection of $\widetilde{\psi}_1^0$ with the geodesic β_i, where we write $\delta_i = \pi/2$ if the geodesics β_i and γ do not intersect. By invariance under ρ we have $\delta_{2p-i} - \pi/2 = -(\delta_i - \pi/2)$.

Following Kerckhoff (see [K]), the derivative at $t = 0$ of the length of $\widetilde{\psi}_1^0$ under our deformation of Ω_0 equals up to a positive constant the sum $\sum a_i \cos\delta_i$. But $0 > \cos\delta_i = -\cos\delta_{2p-i}$ for $1 \le i \le k-1$ and $\cos\delta_i = 0$ otherwise and therefore the derivative of the length of $\widetilde{\psi}_1^0$ under our above deformation of Ω_0 does not vanish. In other words, the map L does not vanish on W. This completes the proof of the corollary. □

Let now again $p \ge 5$ be arbitrary and write $g = (p-1)/2$. Using the above notation, for $M \in \mathcal{T}_{g,3}$ let ψ_i^j be the closed geodesic on the surface PM which is freely homotopic to the compactification of the curve γ_i^j. For $S \in \mathcal{T}_g$ let $\ell_S(\psi_i^j)$ be the length of ψ_i^j. We then obtain a real analytic map Ψ of $\mathcal{T}_g$ into $\mathbf{R}^{3p}$ by mapping S to $\Psi(S) = (\ell_S(\psi_1^0), \dots, \ell_S(\psi_p^2))$.

Theorem B from the introduction is an immediate consequence of the following.

LEMMA 5.4. *The map Ψ is of maximal rank differentiable and injective.*

Proof. Let again ξ_i^j be the tangent of the earthquake path along the closed geodesic ψ_i^j. By the results of Wolpert [W] it suffices to show that the tangent space of $\mathcal{T}_g$ at any point S is spanned by the vectors ξ_i^j.

An arbitrary choice of three points in the complement of the curves ψ_i^j on S defines a surface $M \in \mathcal{T}_{g,3}$. The earthquake path in $\mathcal{T}_g$ induced by ψ_i^j naturally lifts to a path in $\mathcal{T}_{g,3}$. The consideration in the proof of Lemma 5.2 shows that this lift is (up to parametrization and up to possibly moving the punctures) just the earthquake path in S_∞ along $\widetilde{\psi}_i^j \in M$. This implies by Lemma 5.2 and Lemma 5.1 that the tangent space of $\mathcal{T}_g$ at M is spanned by the vectors ξ_i^j and shows that Ψ is of maximal rank differentiable. Since the earthquake paths along the curves γ_i^j parametrize $\mathcal{T}_{g,3}$ the map Ψ is moreover injective. □

The next corollary is an immediate consequence of Lemma 5.3, Lemma 5.4, Proposition 4.8 and the results of Schmutz in [S1].

COROLLARY 5.5. *The surfaces* $S(7;3)$, $S(13;4)$, $S(21;5)$ *and their associated ideal surfaces are maximal.*

We conclude the paper with some remarks about the relation between our triangulation and the structure of the Thurston boundary of Teichmüller space.

Consider for the moment an arbitrary closed surface S. A *geodesic current* for S is a locally finite Borel-measure on the space of unoriented geodesics in the hyperbolic plane $\mathbf{H}^2$ which is invariant under the action of the fundamental group $\pi_1(S)$ of S. The space $\mathcal{C}$ of geodesic currents for S only depends on the topological type of S. There is a bilinear form i on $\mathcal{C}$, the so called *intersection form*, which is continuous with respect to the weak*-topology on $\mathcal{C}$. The subset $\mathcal{L}$ of $\mathcal{C}$ of all geodesic currents μ with vanishing self-intersection $i(\mu,\mu)=0$ is the space of *measured geodesic laminations* and is homeomorphic to $\mathbf{R}^{6g-6}$ [B].

Let $\mathcal{PC}$ and $\mathcal{PL}$ be the projectivization of the space of nonzero geodesic currents and laminations. There is a natural continuous embedding J of the Teichmüller space $\mathcal{T}_g$ into $\mathcal{PC}$ by mapping $M \in \mathcal{T}_g$ to the projectivization $[\lambda_M]$ of its Lebesgue-Liouville current λ_M. The closure of $J(\mathcal{T}_g)$ in $\mathcal{PC}$ is just $J(\mathcal{T}_g) \cup \mathcal{PL}$ [B].

Every simple closed geodesic ψ on S can naturally be viewed as a measured geodesic lamination and hence induces a linear functional on $\mathcal{C}$ via $\mu \to i(\psi,\mu)$. If λ_M is the Lebesgue-Liouville current of a point $M \in \mathcal{T}_g$ in Teichmüller space then $i(\lambda_M,\psi)=\ell_M(\psi)$ is just the M-length of ψ [B]. In particular, the map $M \in \mathcal{T}_g \to i(\lambda_M,\psi)$ is real analytic.

Recall that a collection $\psi_1,\ldots,\psi_k$ of simple closed curves on S *fills up* if every geodesic on S intersects one of the curves ψ_i transversely. This is equivalent to saying that the complement of $\{\psi_1,\ldots,\psi_k\}$ in S consists of a finite collection of connected simply connected regions. If $\psi_1,\ldots,\psi_k$ fills up then for every measured geodesic lamination $\mu \in \mathcal{L}$ the vector $(i(\psi_1,\mu),\ldots,i(\psi_k,\mu)) \in \mathbf{R}^k$ does not vanish. Thus if we denote by $P\mathbf{R}^k$ the real projective space of all lines in $\mathbf{R}^k$ and for $0 \neq x \in \mathbf{R}^k$ by $[x] \in P\mathbf{R}^k$ the line in $\mathbf{R}^k$ through x then the map $A\colon M \in \mathcal{T}_g \to [\ell_M(\psi_1),\ldots,\ell_M(\psi_k)] \in P\mathbf{R}^k$ extends continuously to the Thurston compactification $\mathcal{PL}$ of $\mathcal{T}_g$ by mapping the projective class $[\mu]$ of $\mu \in \mathcal{L}$ to $A([\mu]) = [i(\psi_1,\mu),\ldots,i(\psi_k,\mu)]$. A family $(\psi_1,\ldots,\psi_k)$ of simple closed curves on S is called *parametrizing* for

$\mathcal{PL}$ if the map $[\mu] \in \mathcal{PL} \to A([\mu]) = [i(\psi_1, \mu), \ldots, i(\psi_k, \mu)] \in P\mathbf{R}^k$ is an embedding.

It is also possible to define geodesic currents and measured geodesic laminations for hyperbolic surfaces with cusps. By definition, a measured geodesic lamination of such a surface M with cusps is a compact subset of M which is foliated by geodesics and equipped with a transverse invariant measure.

Let now $p \geq 5$ and let $k \in \{2, \ldots, p-1\}$ be such that k and $k-1$ are prime to p. Denote by S_∞ the ideal surface associated to the triangle surface $S(k;p)$ and let γ_i^j the piecewise geodesics as in Lemma 5.1. If ψ is any closed geodesic in S_∞ then ψ does not disappear in the cusps of S_∞ and hence ψ intersects each of the geodesics γ_i^j transversely in a finite number of points. We denote by $i(\psi, \gamma_i^j)$ the number of intersections of ψ with γ_i^j. Since measured laminations on S_∞ have compact support, intersection of closed geodesics with one of the curves γ_i^j extends to a continuous convex-linear functional $i(\gamma_i^j, \cdot)$ on the space $\mathcal{L}_\infty$ of measured geodesic laminations on S_∞.

We have:

LEMMA 5.6. *The map* $\mu \in \mathcal{L}_\infty \to A(\mu) =$

$$(i(\gamma_1^0, \mu), \ldots, i(\gamma_p^0, \mu), i(\gamma_1^1, \mu), \ldots, i(\gamma_p^1, \mu), i(\gamma_q^2, \mu), \ldots, i(\gamma_p^2, \mu))$$

is an embedding.

Proof. It suffices to show that every closed geodesic ψ is determined by $A(\psi)$. For this consider again the edges α_i^j of the canonical triangulation of S_∞. It follows immediately from our construction that $A(\psi)$ determines uniquely the tuple

$$C(\psi) = (i(\alpha_1^0, \psi), \ldots, i(\alpha_p^0, \psi), i(\alpha_1^1, \psi), \ldots, i(\alpha_p^1, \psi), i(\alpha_1^2, \psi), \ldots, i(\alpha_p^2, \psi))$$

(compare the proof of Lemma 5.1). Thus it is enough to show that we can reconstruct ψ from $C(\psi)$.

The arcs α_j^i define a triangulation of S_∞ into $2p$ triangles with vertices at the cusps and such that each arc is the side of exactly two triangles. Let ψ be any closed geodesic on S_∞ and let T be a triangle from the triangulation with sides $\beta_1, \beta_2, \beta_3$. Write $j_i = i(\beta_i, \psi)$ and assume that $j_1 \geq j_2 \geq j_3$. Since T is contractible in the compactification of S_∞, the total intersection number $j_1 + j_2 + j_3$ of ψ with the boundary of T is even and hence $j_2 + j_3 - j_1$ is even as well. Moreover we have $j_1 \leq j_2 + j_3$. Draw $\frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_2 and $\beta_3, j_2 - \frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_1 and $\beta_2, j_3 - \frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_1 and β_3 in such a way that all these arcs are disjoint. The configuration of these arcs in T is determined up to isotopy by $j_1 \geq j_2 \geq j_3$. But this means that ψ is uniquely determined by $C(\psi)$ (compare the discussion in [FLP]) and hence the lemma follows. $\square$

Recall that a closed curve ψ on S_∞ is *cusp-parallel* if ψ is homotopic to a multiple of a circle surrounding one of the cusps of S_∞. This is equivalent to saying that the infimum of the lengths of all curves in S_∞ which are freely homotopic to ψ is zero (notice that by abuse of notation we call a contractible curve cusp-parallel as well). A closed curve ψ on S_∞ is freely homotopic to a closed geodesic if and only if ψ is not cusp-parallel.

We define now an equivalence relation on the set of all closed curves on S_∞ as follows: Let $\psi, \eta\colon [0,1] \to S_\infty$ be parametrized closed curves. Call ψ, η *equivalent* if there is a subdivision $0 < t_1 < \cdots < t_k < 1$ of $[0,1]$ and for each i there is a cusp-parallel loop γ_i through $\psi(t_i)$ such that η is freely homotopic to $\psi|_{[0,t_1]} \cup \gamma_1 \cdots \cup \gamma_k \cup \psi|_{[t_k,1]}$.

This is easily seen to be an equivalence relation. The equivalence classes of this relation are in $1-1-$correspondence to the free homotopy classes of closed curves on the surface S. We denote the class of ψ by $[\psi]$. For a closed curve ψ on S_∞ and for $i \in \{1, \ldots, p\}$, $j = 0, 1, 2$ define $\mathcal{J}(\psi, \gamma_i^j)$ to be the infimum of the number of intersections with γ_i^j of all curves η equivalent to ψ.

Let ψ_i^j be the closed geodesic on the surface S which is freely homotopic to the compactification of γ_i^j viewed as a curve on S. For every closed geodesic η on S which is different from a multiple of ψ_i^j the number of intersection points between η and ψ_i^j is the infimum $i(\eta, \psi_i^j)$ of the number of intersection points between all curves freely homotopic to η, ψ_i^j.

We have:

LEMMA 5.7. *$\mathcal{J}(\zeta, \gamma_i^j) = i([\zeta], \psi_i^j)$ for every closed curve ζ on S_∞.*

Proof. For every closed curve ζ on S_∞ there is an equivalent curve η such that $\mathcal{J}(\zeta, \gamma_i^j)$ equals the number of intersection points of η with γ_i^j. Now if we compactify S_∞ by adding a point at each cusp, then we obtain a surface M of genus g and η and ζ are freely homotopic on M, γ_i^j is freely homotopic to the curve ψ_i^j. But this means that $\mathcal{J}(\zeta, \gamma_i^j) \geq i([\zeta], \psi_i^j)$.

On the other hand, if ζ is any closed curve on S with a minimal number of intersections with ψ_i^j in its free homotopy class, then we can remove from S three points which do not lie on ζ and such that two of these points lie on ψ_i^j. If we call the resulting surface S_∞ then ζ defines a closed curve ζ_∞ on S_∞, and $i(\zeta,\psi_i^j)$ equals the number of intersection points between ζ_∞ and γ_i^j (where γ_i^j is given as before). This then shows that $\mathcal{J}(\zeta_\infty,\gamma_i^j) \le i(\zeta,\psi_i^j) = i([\zeta_\infty],\psi_i^j)$ $\square$

As an immediate consequence of Lemma 5.6 and Lemma 5.7 we obtain

COROLLARY 5.8. *The curves ψ_i^j on S are parametrizing for $\mathcal{PL}$. In particular, for every $g \ge 2$ there is a family of $6g+3$ free homotopy classes on a closed surface of genus g which is parametrizing for $\mathcal{PL}$.*

REMARK. From [FLP] one immediately obtains a family of $9g-9$ closed curves on a closed surface of genus g which is parametrizing for $\mathcal{PL}$. To my knowledge, the minimal number of simple closed curves with this property is not known.

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