

## 6.2 The Euler class of a group action on the circle

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Now consider the group  $\tilde{\Gamma}_g$  defined by the presentation

$$\tilde{\Gamma}_g = \langle z, a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = z, \quad z a_i = a_i z, \quad z b_i = b_i z \rangle.$$

The central subgroup  $A$  generated by  $z$  turns out to be infinite cyclic so that  $\tilde{\Gamma}_g$  defines a central extension of  $\Gamma_g$  by  $\mathbf{Z}$ , hence an Euler class in  $H^2(\Gamma_g, \mathbf{Z})$ . It is a fact that  $H^2(\Gamma_g, \mathbf{Z})$  is isomorphic with  $\mathbf{Z}$  and that the element that we have just constructed is a generator of this cohomology group. We shall not prove this here but we note that this is related to the fact that a closed oriented surface of genus  $g \geq 1$  has a contractible universal cover and that the cohomology of  $\Gamma_g$  can therefore be identified with the cohomology of the compact oriented surface of genus  $g$  (see [11] for more details).

## 6.2 THE EULER CLASS OF A GROUP ACTION ON THE CIRCLE

We have already met a central extension related to groups of homeomorphisms

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\text{Homeo}}_+(\mathbf{S}^1) \xrightarrow{p} \text{Homeo}_+(\mathbf{S}^1) \longrightarrow 1.$$

The cohomology group  $H^2(\text{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$  has been computed. It is isomorphic to  $\mathbf{Z}$  and a generator is the Euler class of this central extension [50].

Consider now a homomorphism  $\phi$  from some group  $\Gamma$  to  $\text{Homeo}_+(\mathbf{S}^1)$ . Then, we can pull back the previous extension by  $\phi$ . In other words, we consider the set of  $(\gamma, \tilde{f}) \in \Gamma \times \widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  such that  $\phi(\gamma) = p(\tilde{f})$ . This is a group  $\tilde{\Gamma}$  equipped with a canonical projection onto  $\Gamma$  whose kernel is isomorphic to  $\mathbf{Z}$ , i.e.  $\tilde{\Gamma}$  is a central extension of  $\Gamma$  by  $\mathbf{Z}$ . In case  $\phi$  is injective,  $\tilde{\Gamma}$  is just the pre-image of  $\phi(\Gamma)$  under  $p$ , which is the group of lifts of  $\phi(\Gamma)$ . The Euler class of this central extension of  $\Gamma$  is called *the Euler class of the homomorphism  $\phi$*  and denoted by  $eu(\phi) \in H^2(\Gamma, \mathbf{Z})$ . It is obviously a dynamical invariant in the sense that two conjugate homomorphisms  $\phi_1$  and  $\phi_2$  have the same Euler class in  $H^2(\Gamma, \mathbf{Z})$ . Note that it follows from the definition that  $eu(\phi)$  is zero if and only if the homomorphism  $\phi$  lifts to a homomorphism  $\tilde{\phi}: \Gamma \rightarrow \widetilde{\text{Homeo}}_+(\mathbf{S}^1)$  such that  $\phi = p \circ \tilde{\phi}$ .

A few examples are in order. In the case of a single homeomorphism, i.e. when  $\Gamma = \mathbf{Z}$ , we saw that  $H^2(\mathbf{Z}, \mathbf{Z}) = 0$ . Hence the Euler class vanishes and our new invariant is very poor indeed: in particular, it does not detect the rotation number. A similar phenomenon occurs when  $\Gamma$  is free.

If  $\Gamma_g$  is the fundamental group of a closed oriented surface of genus  $g \geq 1$ , we know that  $H^2(\Gamma_g, \mathbf{Z})$  is isomorphic to  $\mathbf{Z}$  so that the Euler class

$eu(\phi)$  in this case is an integer. In [51], Milnor gives an algorithm to compute this number. With the same notation as above, for each  $1 \leq i \leq g$ , choose lifts  $\tilde{a}_i$  and  $\tilde{b}_i$  of  $\phi(a_i)$  and  $\phi(b_i)$ . Now compute the product of commutators  $\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1} \tilde{b}_1^{-1} \dots \tilde{a}_g \tilde{b}_g \tilde{a}_g^{-1} \tilde{b}_g^{-1}$ . Since this homeomorphism is a lift of the identity, it is an integral translation. This amplitude of this translation does not depend on the choices made and is the Euler number  $eu(\phi)$ .

As an explicit example, also computed by Milnor, recall that any closed orientable surface of genus  $g > 1$  can be endowed with a riemannian metric of constant negative curvature. Recall also that the Poincaré upper half space  $\mathcal{H}$  can be equipped with a metric of curvature  $-1$  whose group of orientation preserving isometries is precisely  $\mathrm{PSL}(2, \mathbf{R})$ . Moreover, any complete simply connected riemannian surface of curvature  $-1$  is isometric to  $\mathcal{H}$ . Hence there are embeddings  $\phi$  of the fundamental group  $\Gamma_g$  of a closed oriented surface of genus  $g > 1$  in  $\mathrm{PSL}(2, \mathbf{R})$  such that the corresponding action of  $\Gamma_g$  on  $\mathcal{H}$  is free, proper and cocompact. Since we know that  $\mathrm{PSL}(2, \mathbf{R})$  is a subgroup of  $\mathrm{Homeo}_+(\mathbf{S}^1)$ , we can compute the corresponding Euler number  $eu(\phi)$ . The result of the computation is  $2g - 2$ . Note that each element of  $\phi(\Gamma_g)$  is hyperbolic since the action is free and cocompact so that the rotation number of every element of  $\phi(\Gamma_g)$  is 0. So we are in a situation in which the topological invariant  $eu(\phi)$  is not 0 but the rotation number invariants are trivial; a situation different from the case where  $\Gamma = \mathbf{Z}$ .

### 6.3 BOUNDED COHOMOLOGY AND THE MILNOR-WOOD INEQUALITY

It was observed very early that the Euler class of a homomorphism  $\phi: \Gamma \rightarrow \mathrm{Homeo}_+(\mathbf{S}^1)$  cannot be arbitrary. Milnor and Wood proved the following [51, 71].

**THEOREM 6.1 (Milnor-Wood).** *Let  $\Gamma_g$  be the fundamental group of a closed oriented surface of genus  $g \geq 1$  and  $\phi: \Gamma_g \rightarrow \mathrm{Homeo}_+(\mathbf{S}^1)$  be any homomorphism. Then the Euler number satisfies  $|eu(\phi)| \leq 2g - 2$ .*

*Proof.* We shall not give a complete proof since this result will follow from later considerations but we prove a weaker version. Keeping the previous notation, we know that  $eu(\phi)$  is the translation number of the homeomorphism  $\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1} \tilde{b}_1^{-1} \dots \tilde{a}_g \tilde{b}_g \tilde{a}_g^{-1} \tilde{b}_g^{-1}$ . We also know that the translation number function  $\tau$  is a quasi-homomorphism, i.e. there is some inequality of the form  $|\tau(\tilde{f}_1 \tilde{f}_2) - \tau(\tilde{f}_1) - \tau(\tilde{f}_2)| \leq D$  for some  $D$ . We also know that  $\tau(\tilde{f}^{-1}) = -\tau(\tilde{f})$ . So, if we evaluate  $\tau$  on this element, we get a bound of the form  $|eu(\phi)| \leq (4g - 1)D$ . This is not quite the bound given in the