

## 3.2 Piecewise linear groups

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*fuchsian groups* which are by definition the discrete subgroups of  $\mathrm{PSL}(2, \mathbf{R})$ . These groups come from many parts of mathematics, in particular from number theory. For instance, the modular group  $\mathrm{PSL}(2, \mathbf{Z})$  is fundamental in the study of quadratic forms in two variables over the integers and its action on  $\mathbf{RP}^1$  or on  $\mathcal{H}$  is one of the main tools to understand it. Gauss began its analysis in his famous *Disquisitiones* and the modular group might be the first non-commutative group to have been studied in the history of mathematics. As another example, consider a quadratic form in three variables with integral coefficients and signature  $(+, +, -)$ ; the group of its isometries with integer coefficients is of course a fuchsian group. This was another motivation for Poincaré when he studied these groups [60]. We also want to emphasize that not only the discrete groups of  $\mathrm{PSL}(2, \mathbf{R})$  might be interesting, even from the number theoretical point of view. Examples can be given by taking a number field  $k$  embedded in  $\mathbf{R}$  and looking at the ring of integers  $\mathcal{O}$  in this field (for instance  $\mathbf{Z}[\sqrt{2}]$  in  $\mathbf{Q}(\sqrt{2})$ ). The group  $\mathrm{PSL}(2, \mathcal{O})$  of elements of  $\mathrm{PSL}(2, \mathbf{R})$  with entries in  $\mathcal{O}$  is a very important one (even though it is dense in  $\mathrm{PSL}(2, \mathbf{R})$  if  $k$  is not the field of rational numbers).

### 3.2 PIECEWISE LINEAR GROUPS

Our second example is a much bigger group: the group of piecewise linear homeomorphisms of the circle  $\mathbf{S}^1$ , considered here as  $\mathbf{R}/\mathbf{Z}$ . A homeomorphism  $f$  of the real line  $\mathbf{R}$  is called *piecewise linear* if there is an increasing sequence of real numbers  $x_i$  parametrized by  $i \in \mathbf{Z}$  such that  $\lim_{\pm\infty} x_i = \pm\infty$  and such that the restriction of  $f$  to each interval  $[x_i, x_{i+1}]$  coincides with an affine map. If such a homeomorphism satisfies  $f(x+1) = f(x) + 1$  for all  $x$ , then it induces a homeomorphism of the circle  $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$ . Such a homeomorphism of  $\mathbf{S}^1$  is called a piecewise linear homeomorphism of the circle. Note that, by our definition, we are only considering orientation preserving homeomorphisms of the circle. The collection of these homeomorphisms is a group, denoted by  $\mathrm{PL}_+(\mathbf{S}^1)$ .

Again, this group is acting transitively on the circle so there is not much to say about its orbits... However  $\mathrm{PL}_+(\mathbf{S}^1)$  contains some very interesting subgroups which will provide good examples of some dynamical phenomena on the circle. We shall mention only one of them.

The *Thompson group*, denoted by  $G$ , is a countable subgroup of  $\mathrm{PL}_+(\mathbf{S}^1)$  which has been studied quite a lot recently and deserves more attention. Some of its properties will be mentioned in these notes, in particular as a source of (counter)-examples. To define it, we consider first the group  $\tilde{G}$  consisting

of piecewise linear homeomorphisms  $f$  of  $\mathbf{R}$  which have the following four properties.

- The sequence  $x_i$  can be chosen in such a way that  $x_i$  and  $f(x_i)$  consist of dyadic rational numbers (*i.e.* of the form  $p2^q$ ,  $p, q \in \mathbf{Z}$ ).
- The set of dyadic rational numbers is preserved by  $f$ .
- The derivatives of the restrictions of  $f$  to  $]x_i, x_{i+1}[$  are powers of 2 (*i.e.* of the form  $2^q$ ,  $q \in \mathbf{Z}$ ).
- One has  $f(x + 1) = f(x) + 1$  for all  $x$ .

The elements of  $\tilde{G}$  induce homeomorphisms of the circle  $\mathbf{S}^1 \simeq \mathbf{R}/\mathbf{Z}$ . The collection of these homeomorphisms is the Thompson group  $G$ . Figure 4 shows the graphs of two typical elements of  $G$ .

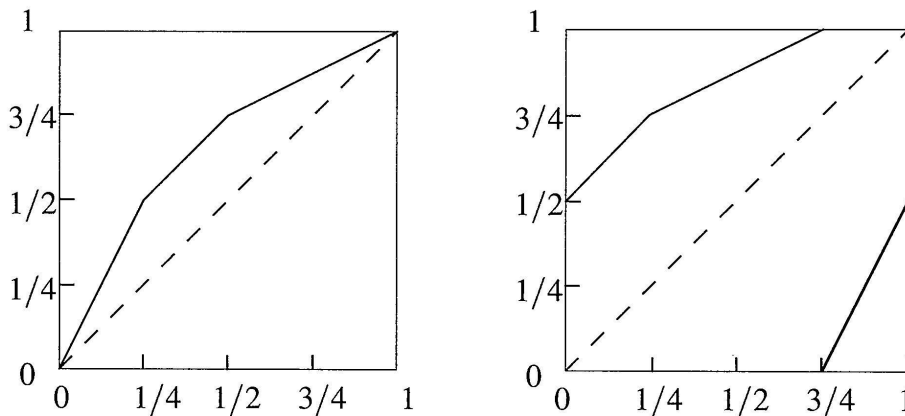


FIGURE 4

Among the nice properties of  $G$ , we mention first the fact that  $G$  is an *infinite finitely presented simple group*. This was the main motivation for Thompson: indeed  $G$  was the first example of such a group (recall that a group is called simple if it contains no proper normal subgroup).

We also mention a connection with the modular group  $\text{PSL}(2, \mathbf{Z})$  acting on  $\mathbf{RP}^1$ . Consider the group of homeomorphisms of  $\mathbf{RP}^1$  which are piecewise- $\text{PSL}(2, \mathbf{Z})$ , *i.e.* for which one can partition  $\mathbf{RP}^1$  as a finite union of intervals with rational endpoints in such a way that on each of these intervals, the homeomorphism coincides with an element of  $\text{PSL}(2, \mathbf{Z})$ . It turns out that there is a homeomorphism  $h$  from  $\mathbf{R}/\mathbf{Z}$  to  $\mathbf{RP}^1$  mapping the dyadic points in  $\mathbf{R}/\mathbf{Z}$  to the rational points of  $\mathbf{QP}^1$  and conjugating the Thompson group  $G$  with this group of piecewise- $\text{PSL}(2, \mathbf{Z})$ !

Somehow, we could say that  $G$  sits inside  $\text{PL}_+(\mathbf{S}^1)$  like a fuchsian group sits inside  $\text{PSL}(2, \mathbf{R})$ . For more information concerning this group, see [13, 28].