4. A Lie algebra representation

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4. A Lie algebra representation

Let M be a projective R-module of rank two. Let $G = \mathrm{Aut}_R(M)$ and let $\mathfrak{g} = \mathrm{End}_R(M)$ viewed as a Lie algebra over R.

The group G acts on the right on $\mathrm{Sym}_R(M^*)$ by algebra automorphisms via

$$(F\sigma)(\mathbf{x}) = F(\sigma\mathbf{x})$$

for $F \in \mathrm{Sym}_R(M^*)$ and $\sigma \in G$. Taking the formal derivative at the origin of the associated map

$$G \to \mathrm{Aut}_{R-\mathrm{alg}}(\mathrm{Sym}_R(M^*))$$

we get a representation of Lie algebras

$$\rho\colon \mathfrak{g} \longrightarrow \mathrm{Der}_R(\mathrm{Sym}_R(M^*)). \tag{22}$$

The action of G preserves the homogeneous components $\mathrm{Sym}^n_R(M^*)$ and also the submodule $S^n(M^*)$ of Gaussian forms. The same is true for the Lie algebra action of $\mathfrak{g}$.

We shall compute the action of $\mathfrak{g}$ on $S^n(M^*)$ explicitly:

Lemma 4.1. *Let $F \in S^n(M^*)$ and let T be the associated n-linear form. Then*

$$\rho(g)(F)(\mathbf{x}) = nT(g\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$$

for all $g \in \mathfrak{g}$.

Proof. To compute the derivative of $G \to \mathrm{Aut}_R(S^n(M^*))$, we extend the scalars to the "dual numbers" $R[\epsilon]/(\epsilon^2)$. Using the symmetry of T we have

$$F((1+g\epsilon)\mathbf{x}) = F(\mathbf{x}) + nT(g\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})\,\epsilon,$$

which proves our assertion. $\square$

Let C/R be a quadratic algebra in the sense of Section 2 and let M be an invertible C-module. Then we have a natural map $C \to \mathrm{End}_R(M)$ and we can restrict the representation ρ to C. Note that when R is a field and C is an étale quadratic algebra then the image of C is a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}$.

Comparing (22) with equation (21), we see that the C-module structure on $S^3_C(M^*)$ is related to the Lie algebra action by

$$cF = \tfrac{1}{3}\rho(c)F. \tag{23}$$

We will make this explicit in a special case that we need:

LEMMA 4.2. *Let $F \in S^3(M^*)$ be a binary cubic form over a field K of characteristic not 2 or 3. Let q_F be its determining form, and $C = C^+(q_F)$ its even Clifford algebra. Let x_1, x_2 be coordinates on the vector space M with respect to a basis $\mathbf{m}_1, \mathbf{m}_2$. Let*

$$\tau = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1 \in C = C^+(q_F).$$

Note that $\tau^2 = D$ is the discriminant of q_F. Then

$$\rho(\tau) = \frac{\partial q_F}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial q_F}{\partial x_1}\frac{\partial}{\partial x_2},$$

acting on forms of any degree.

Proof. As we have seen,

$$q_F(x_1\mathbf{m}_1 + x_2\mathbf{m}_2) = Px_1^2 + Qx_1x_2 + Rx_2^2,$$

where $P = a_1^2 - a_0a_2$, $Q = a_1a_2 - a_0a_3$, and $R = a_2^2 - a_1a_3$. By direct computation in the Clifford algebra C, we see that

$$\tau\mathbf{m}_1 = Q\mathbf{m}_1 - 2P\mathbf{m}_2$$
$$\tau\mathbf{m}_2 = 2R\mathbf{m}_1 - Q\mathbf{m}_2.$$

Since $\rho(c)$ is a derivation of $\mathrm{Sym}_R(M^*)$, we have

$$\rho(c) = \rho(c)(x_1)\frac{\partial}{\partial x_1} + \rho(c)(x_2)\frac{\partial}{\partial x_2}.$$

Thus $\tau(x_1\mathbf{m}_1 + x_2\mathbf{m}_2) = (Qx_1 + 2Rx_2)\mathbf{m}_1 - (2Px_1 + Qx_2)\mathbf{m}_2$, which gives $\rho(\tau)(x_1) = \partial q_F/\partial x_2$ and $\rho(\tau)(x_2) = -\partial q_F/\partial x_1$. □

COROLLARY 4.3. *$\rho(\tau)q_F = 0$ and*

$$\rho(\tau)F = \begin{vmatrix} \partial F/\partial x_1 & \partial F/\partial x_2 \\ \partial q_F/\partial x_1 & \partial q_F/\partial x_2 \end{vmatrix} = 3G_F, \tag{24}$$

where G_F is as in (5).

REMARK 4.4. If we further assume that C is an étale algebra, then as we have remarked, ρ maps C onto a Cartan subalgebra of $\mathrm{End}_K(M) \sim \mathfrak{gl}(2, K)$. This algebra decomposes as

$$\mathfrak{h}_C = \mathfrak{z} \oplus \mathfrak{h}'_C$$

where the first factor is the center, consisting of scalar matrices, and the second factor is the intersection $\mathfrak{h}_C \cap \mathfrak{sl}(2, K)$, consisting of matrices of trace 0. As the formulas in the proof of the preceding lemma show that τ acts on M with trace 0, we see that $\mathfrak{h}'_C = K\tau$.

THEOREM 4.5. *Let C/R be a quadratic algebra such that $C \otimes K$ is étale over K. Let M be a projective rank-one C-module and let $F \in S^3(M^*)$ be such that the determining mapping q_F is not 0. Then the following conditions are equivalent:*

(a) *F is a C-form*

(b) *$(M, q_F, \mathcal{D}(M))$ is of type C*

(c) *$\rho(c)\rho(\bar{c})F = 9n(c)F$ for all $c \in C$.*

Proof. (a)$\Rightarrow$(b). If T is the trilinear form attached to F, then, using the symmetry of $T(c\mathbf{x}, \mathbf{y}, \mathbf{z})$, we have

$$\begin{aligned} q_F(c\mathbf{x}) &= \wedge^2 T(c\mathbf{x}, -, -) \\ &= \wedge^2 (T(\mathbf{x}, c-, -)) \\ &= n(c) \wedge^2 (T(\mathbf{x}, -, -)) \\ &= n(c) q_F(\mathbf{x}), \end{aligned}$$

which proves the claim. In fact, this implication does not depend on $C \otimes K$ being étale.

It is enough to prove the theorem for the case where $R = K$ is a separably closed field. We can assume in this case $C = K[\sigma]$ with σ satisfying $\sigma^2 = 1$. We will make these assumptions for the rest of the proof.

(b)$\Rightarrow$(c). Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis of M over K with $\sigma\mathbf{m}_1 = \mathbf{m}_1$ and $\sigma\mathbf{m}_2 = -\mathbf{m}_2$. With respect to this basis, the form q_F, being of type C, must have the shape

$$q_F(\mathbf{x}) = \alpha x_1 x_2,$$

where $\alpha \neq 0$. To see that this is so, note that because q_F is of type C, we have $q_F(\sigma\mathbf{m}_1) = n(\sigma)q_F(\mathbf{m}_1) = -q_F(\mathbf{m}_1)$, which shows that $q_F(\mathbf{m}_1) = 0$. One sees similarly that $q_F(\mathbf{m}_2) = 0$. Then the coefficients of $F(\mathbf{x}) = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3$ satisfy the relations: $a_1^2 - a_0a_2 = 0$, $a_1a_2 - a_0a_3 = \alpha$ and $a_2^2 - a_1a_3 = 0$. Since $\alpha \neq 0$, it follows at once that $a_1 = a_2 = 0$, so F is of the form $F(\mathbf{x}) = \lambda x_1^3 + \mu x_2^3$. Since $q_F \neq 0$ (in fact nondegenerate under the étaleness hypothesis), the algebra C can be identified with the even Clifford algebra $C^+(M, q_F, \mathcal{D}(M))$ by Proposition 2.8. Under that identification we have $\tau = \alpha\sigma$, where τ is defined as in Lemma 4.2. From that lemma we get $\rho(\sigma) = x_1\partial/\partial x_1 - x_2\partial/\partial x_2$, which can be seen directly, since both sides agree on x_1, x_2. Hence $\rho(\sigma)(x_1^{3-i}x_2^i) = (3-2i)x_1^{3-i}x_2^i$. In particular, for $F(\mathbf{x}) = \lambda x_1^3 + \mu x_2^3$ we have

$$\rho(\sigma)\rho(\bar{\sigma})F = -\rho(\sigma)^2 F = -9F = 9n(\sigma)F.$$

The more general identity $\rho(c)\rho(\overline{c})F = 9n(c)F$ for $c \in C$ follows from this particular case by noting that, from Lemma 4.1, $\rho(1)F = 3F$.

(c)$\Rightarrow$(a). Suppose that $\rho(\sigma)^2 F = 9F$. Then F must have the form $F = \lambda x_1^3 + \mu x_2^3$. This is because, as we saw in the discussion above, the monomials $x_1^{3-i}x_2^i$ are eigenvectors for the operator $\rho(\sigma)^2$ with eigenvalue $(3-2i)^2$, which equals 9 only for $i = 0$ and $i = 3$. Hence the associated trilinear form is $T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \lambda x_1 y_1 z_1 + \mu x_2 y_2 z_2$. Thus $T(\sigma\mathbf{x}, \mathbf{y}, \mathbf{z}) = \lambda x_1 y_1 z_1 - \mu x_2 y_2 z_2$, which is visibly symmetric in $\mathbf{x}, \mathbf{y}, \mathbf{z}$. □

REMARK 4.6. It is interesting to notice that the syzygy (6) can be recovered from Part (c) of Theorem 4.5. Assume for simplicity that $R = K$ is a field and C is an étale K-algebra. Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis of M. Let $\tau = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1 \in C = C^+(q_F)$ as in Lemma 4.2. As we noted in Remark 4.4, τ generates the trace 0 part of the Cartan subalgebra defined by C. Using the derivation property and Corollary 4.3, we see $\rho(\tau)(G_F^2 - DF^2) = (2/3)(\rho(\tau)^2 F - 9DF)G_F$. From the above theorem, $\rho(\tau)^2 F = 9DF$, so this is 0. On the other hand, $\rho(\tau)q_F = 0$, also by Corollary 4.3, which implies that $\rho(\tau)q_F^3 = 0$. Hence both q_F^3 and $G_F^2 - DF^2$ lie in the subspace on weight 0 (for the action of the Cartan subalgebra $\mathfrak{h}'_C \subset \mathfrak{sl}(2, K)$) of $S^6(M^*)$. As $S^6(M^*)$ is an irreducible representation of $\mathfrak{sl}(2, K)$, this is one-dimensional. Hence q_F^3 and $G_F^2 - D_F F^2$ differ by a constant multiple. A priori, this constant could depend on F (e.g., D). That this is not so can be seen by noting that both sides are of the same degree in the coefficients of F.

COROLLARY 4.7. *Let M be a projective R-module of rank 2, and let $F \in S^3(M^*)$.*

(i) *Let $C = C^+(M, q_F, \mathcal{D}(M))$ and suppose that $C \otimes K$ is étale, and that q_F is primitive. Then F is a C-form.*

(ii) *If F is a C-form for a quadratic R-algebra C and $(M, q_F, \mathcal{D}(M))$ is primitive, then $C \cong C^+(M, q_F, \mathcal{D}(M))$.*

Proof. (i) By Proposition 2.8, $(M, q_F, \mathcal{D}(M))$ is of type C. We conclude by Theorem 4.5.

(ii) If F is a C-form, then by Theorem 4.5, the quadratic mapping $(M, q_F, \mathcal{D}(M))$ is type C. But assuming furthermore that $(M, q_F, \mathcal{D}(M))$ is primitive, we see that C is isomorphic with $C^+(M, q_F, \mathcal{D}(M))$ by Proposition 2.8. □

LEMMA 4.8. *Suppose that $C \otimes K$ is étale over K and let (M, F) and (M', F') be cubic C-forms. Assume that the determining mappings $q_F, q_{F'}$ are nonzero. Then every R-linear isomorphism $f\colon (M, F) \to (M', F')$ is either C-linear or C-sesquilinear.*

Proof. The map f will induce an isomorphism of determining quadratic mappings of type C. We conclude by Proposition 2.3. □

5. STRUCTURE OF THE CUBIC C-FORMS

We shall describe below the C-module structure of $S^3_C(M^*)$ and the corresponding C-isomorphism classes.

THEOREM 5.1. *Let M be a rank-one projective C-module. For each $\phi \in \mathrm{Hom}_C(M_C^{\otimes 3}, C^*)$ we define a cubic form by $F_\phi(\mathbf{x}) = \phi(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x})(1)$. Then*

(i) *The correspondence $\phi \mapsto F_\phi$ is an isomorphism of C-modules $\mathrm{Hom}_C(M_C^{\otimes 3}, C^*) \to S^3_C(M^*)$.*

(ii) *The determining mapping q_{F_ϕ} is primitive if and only if ϕ is an isomorphism.*

(iii) *Two cubic C-forms F and F_1 on M are equivalent over C if and only if there exists $c \in C^\times$ such that $F_1 = c^3 F$.*

Proof. (i) This is a restatement of Proposition 3.7. The map $\phi \mapsto F_\phi$ is a C-isomorphism by definition of the structure of C-module on $S^3_C(M^*)$ in Section 3.

(ii) It is enough to prove our assertion locally, so we assume that M is free over C. Write $M = C\mathbf{m}$ for some $\mathbf{m} \in M$. Let $\lambda = \phi(\mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m})$. Then we have $\phi(x\mathbf{m} \otimes y\mathbf{m} \otimes z\mathbf{m}) = \lambda(xyz)$. Let $\beta(y\mathbf{m}, z\mathbf{m}) = \lambda(yz)$ and observe that λ is a basis of C^* over C if and only if the symmetric bilinear form β is unimodular. We have

$$\begin{aligned} q_{F_\phi}(x\mathbf{m}) &= n(x) q_{F_\phi}(\mathbf{m}) \\ &= n(x) \wedge^2 \beta\,. \end{aligned}$$

It follows from this equality that q_{F_ϕ} is primitive if and only if β is unimodular, that is, if and only if ϕ is an isomorphism.