2. Flatspaces

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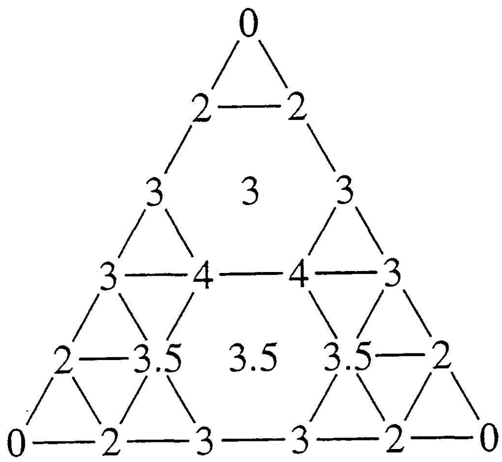
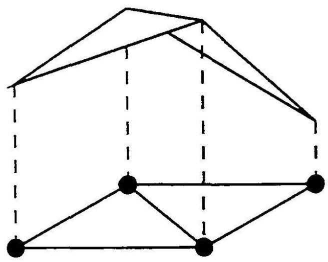


2. Flatspaces

We can consider a hive as a graph over the hive triangle. At each hive vertex we use the label as the height. We then extend these heights to a graph over the entire hive triangle by using linear interpolation over each small triangle. A rhombus inequality now says that the graph over the rhombus cannot bend up across the middle line.



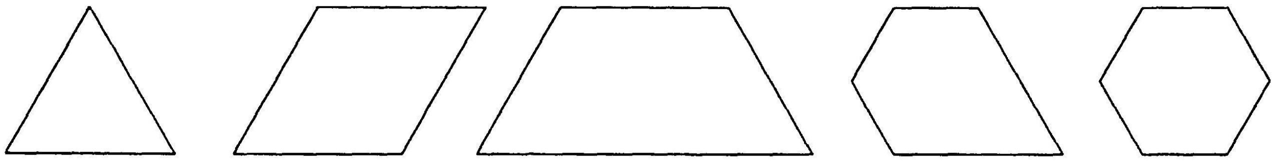
In this way the graph becomes the surface of a convex mountain. The graph is flat (but not necessarily horizontal) over a rhombus if and only if the rhombus inequality is satisfied with equality.

We define a *flatspace* to be a maximal connected union of small triangles such that any contained rhombus is satisfied with equality. The flatspaces split the hive triangle up in disjoint regions over which the mountain is flat. The flatspaces of the hive in Example 2 consist of two hexagons and 13 small triangles.

Flatspaces have a number of nice properties. We will list the ones we need below. Since all of these are straightforward to prove directly from the definitions, we will simply give intuitive reasons for them.

1. *Flatspaces are convex.* This is clear since they lie under intersections of a convex mountain with a (convex) plane.

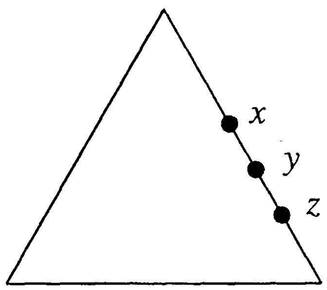
2. *All flatspaces have one of the following five shapes (up to rotations and different side lengths)*:



These are the only convex shapes that can be constructed from small triangles.

3. *A side of a flatspace is either on the border of the big hive triangle, or it is also a side of a neighbor flatspace.* In other words, a side of one flatspace can't be shared between several neighbor flatspaces. This again follows from the convexity of the mountain described above.

Given a labeling $b \in \mathbf{R}^B$, let x, y, z be labels of consecutive border vertices on the same side of the big hive triangle (in any direction).



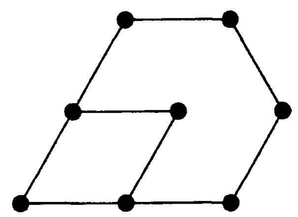
If b is the border of a hive, then the rhombus inequalities imply that $y-x \geq z-y$, although more inequalities are needed to guarantee that $b \in \rho(C)$. We will say that b is *regular* if we always have $y-x > z-y$, when x, y, z are chosen in this way. When a border comes from a triple of partitions, it is regular exactly when each partition is strictly decreasing.

4. *If the border of a hive is regular then no flatspace has a side of length ≥ 2 on the border of the big hive triangle.* In fact, if the labels x, y, z above are on a flatspace side, then $y-x = z-y$.

Given a hive, a non-empty subset $S \subset H - B$ is called *increasable* if the same small positive amount can be added to the labels of all hive vertices in S, such that the labeling is still a hive.

5. *The interior vertices of a hexagon-shaped flatspace form an increasable subset.* Proving this is a matter of checking that each rhombus inequality still holds after adding a small enough amount to the labels of these vertices. Only rhombi that are already flat need to be considered, since for all others there is some "slack to cut".

Note that the corresponding statements for flatspaces of other shapes are false. The reason is that all other shapes have at least one sharp corner (with a 60° angle). Lifting the interior vertex closest to a sharp corner is prohibited by the inequality of the rhombus in that corner.



Note also that the sharp corners of a flatspace of any shape are endpoints of its longest sides.

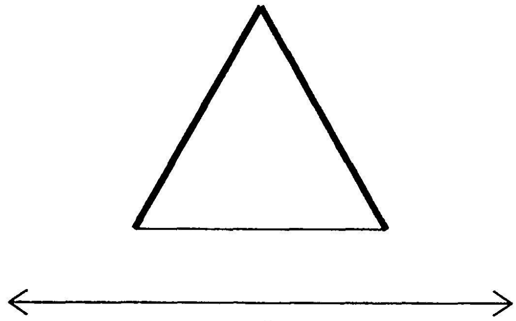
PROPOSITION 1. *If a hive with regular border has no increasable subsets, then its flatspaces consist of small triangles and small rhombi.*

Proof. Otherwise some flatspace has a side of length ≥ 2. This follows because the only types of flatspaces that have all sides of length one are small triangles, rhombi, and small hexagons, and the later do not occur by property 5.

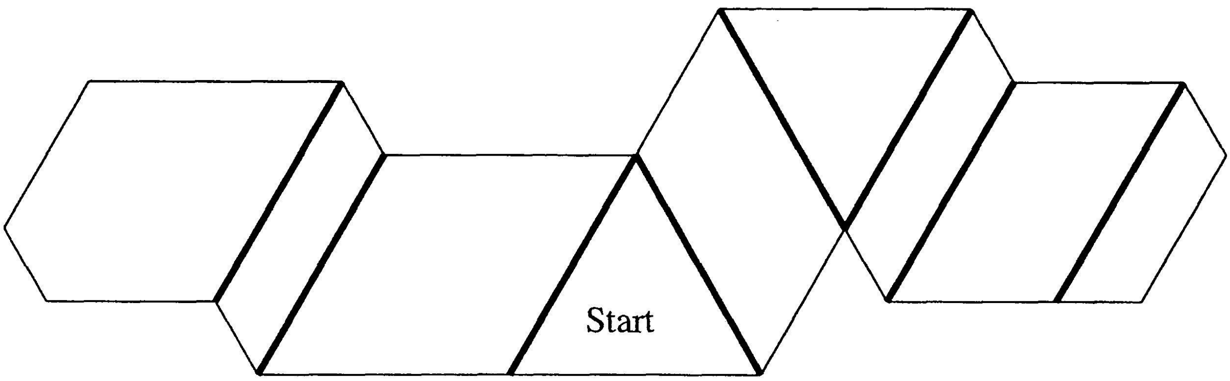
Let m be the maximal length among all sides of flatspaces. We will proceed by constructing a region consisting of flatspaces with a side of length m, such that the interior hive vertices of the region is an increasable subset. The crucial point is to avoid sharp corners pointing out from the region, since otherwise we would get the same problems as with the pentagon above. We need $m \geq 2$ to be sure that interior vertices exist.

Start by taking any flatspace having a side of length m, and mark this side. In the pictures this is shown by making the side thick. Then choose (and fix) a line crossing (the extension of) the marked side in an angle of 60° and call it the moving direction. If the flatspace is a triangle or a parallelogram, we furthermore mark an additional side. For a triangle, this is the other side not parallel to the moving direction, while for a parallelogram we mark the side opposite the one already marked.

We construct a region, starting with the chosen flatspace. This region will initially have one or two marked sides, depending on the shape of the chosen



flatspace. As long as the region has a marked side on its outer border, the flatspace on the opposite side is added to the region. Note that this flatspace is well defined by property 3, since regularity prevents any marked edges from being on the border of the big hive triangle. If the new flatspace is a triangle, we mark its unmarked side which is not parallel to the moving direction. If the new flatspace is a parallelogram, we mark the side opposite the old marked side. If it is not a triangle or parallelogram, we don't mark any new sides.



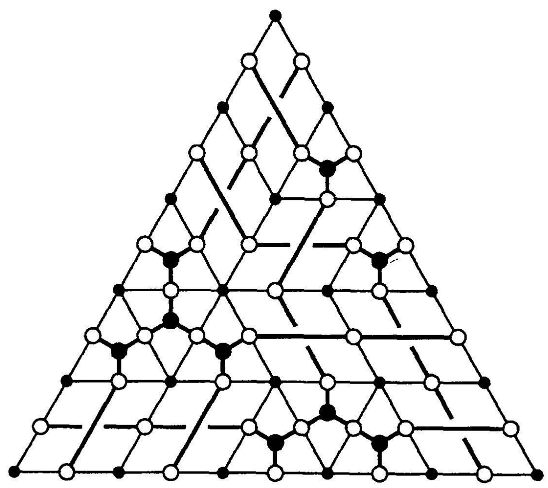
Since the region always grows along the moving direction, it will never go in loops. Now since no marked edges can ever reach the border of the big hive triangle, the described process will stop. Notice that by construction of the region, each included flatspace has enough of its longest sides marked, that all sharp corners are endpoints of marked sides. Since the final region has no marked sides on its outer border, this means that it can't have any sharp corners pointing out.

We claim that the inner vertices of the final region form an increasable subset. If not, some small rhombus satisfied with equality in the region has more obtuse than acute vertices on the region border. If any flat rhombus has both of its obtuse vertices on the border, then it follows, using convexity of the flatspace containing the rhombus, that one of the acute vertices is a sharp corner of the region. On the other hand, if a flat rhombus has one obtuse vertex and no acute vertices on the border, one can deduce, using property 3, that two marked sides must meet in a $120°$ angle at the obtuse vertex on the border. However, the construction never introduces marked sides that meet in this angle.

We have established that the presence of any flatspace which is not a small triangle or rhombus gives rise to an increasable subset. This completes the proof. □

3. Small Flatspaces

Let h be a hive, all of whose flatspaces are small triangles or small rhombi. We construct a graph G from h as follows. G has one fat black vertex in the middle of each small triangle flatspace. In addition there is one circle vertex on every flatspace side. Each fat vertex is connected to the three vertices on the sides of its triangle, and the two circle vertices on opposite sides of a flat rhombus are connected. This graph is topologically equivalent to the reduced honeycomb tinkertoy of Knutson and Tao.



LEMMA 1. *If h is a corner of its hive polytope $\rho^{-1}(\rho(h)) \cap C$, then G is acyclic.*

Proof. Suppose G has a non-trivial loop, and give this loop an orientation. Each hive vertex then has a well defined winding number, which is the number of times the loop goes around this vertex, counted positive in the counter clockwise direction. Note that the winding number is zero for each border vertex, and that some winding numbers are non-zero if the loop is not trivial.

For each $r \in \mathbf{R}$, let $h_r \in \mathbf{R}^H$ be the labeling which maps each hive vertex to the label of h at the vertex plus r times the winding number of the vertex. We will show that h_r is a hive for $r \in (-\epsilon, \epsilon)$, for a suitable $\epsilon > 0$. This