

3. A parametric ideal non-symmetric solution of the Tarry-Escott problem of degree four

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3. A PARAMETRIC IDEAL NON-SYMMETRIC SOLUTION OF THE TARRY-ESCOTT PROBLEM OF DEGREE FOUR

We will now solve the equations

$$(19) \quad a_1 + a_2 + a_3 + a_4 + a_5 = b_1 + b_2 + b_3 + b_4 + b_5,$$

$$(20) \quad a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2,$$

$$(21) \quad a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = b_1^3 + b_2^3 + b_3^3 + b_4^3 + b_5^3,$$

$$(22) \quad a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 = b_1^4 + b_2^4 + b_3^4 + b_4^4 + b_5^4,$$

so as to get an ideal non-symmetric solution of the Tarry-Escott problem of degree four. We write

$$(23) \quad \begin{aligned} a_1 &= 2px - (\xi + \eta)y, & b_1 &= 2px + \eta y, \\ a_2 &= 2qx + \eta y, & b_2 &= 2qx - (\xi + \eta)y, \\ a_3 &= rx, & b_3 &= rx + \zeta y, \\ a_4 &= sx - \zeta y, & b_4 &= sx, \\ a_5 &= \zeta y, & b_5 &= -\zeta y. \end{aligned}$$

We will first choose p, q, r, s, ξ, η and ζ such that equations (19), (20) and (21) are identically satisfied for all values of x and y . In the equation obtained from (22) by substituting the values of a_i, b_i as given above, the coefficients of x^4 and y^4 on the two sides are equal, and we will choose p, q, r, s, ξ, η and ζ so as to satisfy the additional condition that the coefficient of xy^3 also becomes equal on both sides of this equation. Thus, equation (22) would reduce to an equation containing only the terms x^3y and x^2y^2 and accordingly it can be readily solved for x and y . These values of x and y together with the already suitably chosen values of p, q, r, s, ξ, η and ζ substituted in (23) will give a solution of equations (19), (20), (21) and (22).

When a_i, b_i are defined by (23), we observe that equation (19) is identically satisfied. Substituting the values of a_i, b_i in (20), we note that this equation will also be identically satisfied for all values of x and y if the following condition is satisfied:

$$(24) \quad 2(\xi + 2\eta)(p - q) + \zeta(r + s) = 0.$$

Next, we substitute the values of a_i, b_i as given by (23) in equation (21) and observe that the coefficients of x^3 and y^3 on both sides are equal. Equating the coefficients of x^2y and xy^2 on both sides of this equation, we get the following two conditions:

$$(25) \quad 4(\xi + 2\eta)(p^2 - q^2) + \zeta(r^2 + s^2) = 0,$$

and

$$(26) \quad 2\xi(\xi + 2\eta)(p - q) - \zeta^2 r + \zeta^2 s = 0.$$

Finally, in the equation obtained by substituting the values of a_i , b_i in (22), we equate, as already discussed, the coefficients of xy^3 on both sides to get the additional condition:

$$(27) \quad 2(\xi^3 + 3\xi^2\eta + 3\xi\eta^2 + 2\eta^3)(p - q) + \zeta^3(r + s) = 0.$$

We now proceed to solve equations (24), (25), (26) and (27). Equations (24) and (27) may be considered as two linear equations in the two linear variables $(p - q)$ and $(r + s)$, and they will be consistent only if ξ , η and ζ satisfy the condition

$$(\xi + 2\eta)(\xi^2 + \xi\eta + \eta^2 - \zeta^2) = 0.$$

Taking $(\xi + 2\eta) = 0$ leads to trivial solutions, so we will choose ξ , η and ζ such that

$$(28) \quad \xi^2 + \xi\eta + \eta^2 - \zeta^2 = 0.$$

The complete solution of (28) is readily found to be

$$(29) \quad \xi = 2mn - m^2, \quad \eta = m^2 - n^2, \quad \zeta = m^2 - mn + n^2.$$

Next, we solve equations (24) and (26) for r and s , and substitute their values in equation (25) which now has a linear factor $(p - q)$ that can be ignored and then equation (25) is readily seen to be satisfied if we choose p and q as follows:

$$(30) \quad \begin{aligned} p &= \xi^3 + 2\xi^2\eta + \xi\zeta^2 + 2\eta\zeta^2 - 2\zeta^3, \\ q &= \xi^3 + 2\xi^2\eta + \xi\zeta^2 + 2\eta\zeta^2 + 2\zeta^3. \end{aligned}$$

With these values of p and q , we immediately get

$$(31) \quad \begin{aligned} r &= -4\xi^2\zeta - 8\xi\eta\zeta + 4\xi\zeta^2 + 8\eta\zeta^2, \\ s &= 4\xi^2\zeta + 8\xi\eta\zeta + 4\xi\zeta^2 + 8\eta\zeta^2. \end{aligned}$$

Thus, when ξ , η , ζ are defined by (29), and p , q , r and s are given by (30) and (31), equations (24), (25), (26) and (27) are all satisfied. With these values of p, q, r, s, ξ, η and ζ , equation (22) reduces, on removing the factor $64x^2y\zeta^3(\xi + 2\eta)$, to

$$\begin{aligned} &(6\xi^6 + 24\xi^5\eta + 24\xi^4\eta^2 - 12\xi^4\zeta^2 - 48\xi^3\eta\zeta^2 - 48\xi^2\eta^2\zeta^2 - 2\xi^2\zeta^4 \\ &- 8\xi\eta\zeta^4 - 8\eta^2\zeta^4 + 8\zeta^6)x - (3\xi^4 + 6\xi^3\eta - 3\xi^2\zeta^2 - 6\xi\eta\zeta^2)y = 0. \end{aligned}$$

Thus, equation (22) will be satisfied if we choose

$$\begin{aligned}
 (32) \quad x &= 3\xi^4 + 6\xi^3\eta - 3\xi^2\zeta^2 - 6\xi\eta\zeta^2, \\
 y &= 6\xi^6 + 24\xi^5\eta + 24\xi^4\eta^2 - 12\xi^4\zeta^2 - 48\xi^3\eta\zeta^2 \\
 &\quad - 48\xi^2\eta^2\zeta^2 - 2\xi^2\zeta^4 - 8\xi\eta\zeta^4 - 8\eta^2\zeta^4 + 8\zeta^6.
 \end{aligned}$$

A solution of equations (19), (20), (21) and (22) can now be obtained in terms of the parameters m and n by taking ξ , η and ζ as given by (29), substituting these values of ξ , η and ζ in (30), (31) and (32) to obtain p , q , r , s , x and y in terms of m and n , and then substituting the values of p , q , r , s , ξ , η , ζ , x and y in (23). The solution so obtained may, after simplification and removal of common factors, be written explicitly in terms of the arbitrary parameters m and n as follows:

$$\begin{aligned}
 (33) \quad a_1 &= 12m^7n - 37m^6n^2 + 24m^5n^3 + 12m^4n^4 - 20m^3n^5 \\
 &\quad + 15m^2n^6 - 18mn^7 + 8n^8, \\
 a_2 &= 10m^7n - 30m^6n^2 + 54m^5n^3 - 13m^4n^4 - 48m^3n^5 \\
 &\quad + 45m^2n^6 - 14mn^7, \\
 a_3 &= 4m^8 + 6m^7n - 28m^6n^2 + 8m^5n^3 + 66m^4n^4 \\
 &\quad - 128m^3n^5 + 112m^2n^6 - 48mn^7 + 8n^8, \\
 a_4 &= 4m^8 - 12m^7n + 35m^6n^2 - 55m^5n^3 + 66m^4n^4 \\
 &\quad - 65m^3n^5 + 49m^2n^6 - 30mn^7 + 8n^8, \\
 a_5 &= -4m^8 + 14m^7n - 27m^6n^2 + 55m^5n^3 - 80m^4n^4 \\
 &\quad + 81m^3n^5 - 49m^2n^6 + 14mn^7, \\
 b_1 &= -4m^8 + 14m^7n - 22m^6n^2 + 10m^5n^3 + 67m^4n^4 \\
 &\quad - 140m^3n^5 + 113m^2n^6 - 46mn^7 + 8n^8, \\
 b_2 &= 4m^8 + 8m^7n - 45m^6n^2 + 68m^5n^3 - 68m^4n^4 \\
 &\quad + 72m^3n^5 - 53m^2n^6 + 14mn^7, \\
 b_3 &= 20m^7n - 55m^6n^2 + 63m^5n^3 - 14m^4n^4 - 47m^3n^5 \\
 &\quad + 63m^2n^6 - 34mn^7 + 8n^8, \\
 b_4 &= 2m^7n + 8m^6n^2 - 14m^4n^4 + 16m^3n^5 - 16mn^7 + 8n^8, \\
 b_5 &= 4m^8 - 14m^7n + 27m^6n^2 - 55m^5n^3 + 80m^4n^4 \\
 &\quad - 81m^3n^5 + 49m^2n^6 - 14mn^7.
 \end{aligned}$$

We may apply Frolov's theorem to the above solution to obtain other non-symmetric solutions. For instance, an arbitrary constant K can be added to all the terms a_i , b_i , $i = 1, 2, 3, 4, 5$.

As a numerical example, taking $m = 3$, $n = 1$, we get, on suitable re-arrangement, the following solution:

$$\begin{aligned} (-1659)^r + 1406^r + 2784^r + 4025^r + 5915^r \\ = (-1675)^r + 1659^r + 2366^r + 4256^r + 5865^r, \end{aligned}$$

where $r = 1, 2, 3, 4$. Adding the constant 1676 to all the terms, we get the following solution in positive integers:

$$17^r + 3082^r + 4460^r + 5701^r + 7591^r = 1^r + 3335^r + 4042^r + 5932^r + 7541^r,$$

where $r = 1, 2, 3, 4$.

4. THE DIOPHANTINE SYSTEM $\sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r$, $r = 1, 2, 3, 4, 6$

We will now state the theorem given by Gloden [7, p.24] to which a reference has already been made in the introduction and then apply it to obtain a parametric solution of this diophantine system.

THEOREM 4.1. *If*

$$\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \quad r = 1, 2, \dots, k$$

then

$$\sum_{i=1}^{k+1} (a_i + t)^r = \sum_{i=1}^{k+1} (b_i + t)^r, \quad r = 1, 2, \dots, k, k+2,$$

where

$$t = -\left(\sum_{i=1}^{k+1} a_i\right)/(k+1).$$

As we have already obtained, in the preceding section, a parametric solution of $\sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r$, $r = 1, 2, 3, 4$, a direct application of the above theorem gives a parametric solution of $\sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r$, $r = 1, 2, 3, 4$ and 6. We multiply the $(a_i + t)$, $(b_i + t)$, $i = 1, 2, 3, 4, 5$ by 5 to cancel out denominators,