

## 2. The complete ideal symmetric solution of the Tarry-Escott problem of degree four

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$$(5) \quad \sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \quad r = 1, 2, \dots, k, k+2.$$

This follows from a theorem given by Gloden [7, p.24]. Symmetric ideal solutions cannot be used effectively for this purpose as the solutions obtained by applying this theorem hold trivially either for all odd or for all even values of  $r$  according as  $k$  is odd or even.

In this paper, we will obtain the complete ideal symmetric solution of the Tarry-Escott problem of degree four as well as a parametric ideal non-symmetric solution of this problem. We shall use the non-symmetric solution to obtain a parametric solution of the system of equations

$$(6) \quad \sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r, \quad r = 1, 2, 3, 4, 6.$$

Parametric solutions of the system of equations (6) have not been obtained earlier.

## 2. THE COMPLETE IDEAL SYMMETRIC SOLUTION OF THE TARRY-ESCOTT PROBLEM OF DEGREE FOUR

To obtain the complete ideal symmetric solution of degree four, we have to obtain a solution of the system of equations

$$(7) \quad \sum_{i=1}^5 a_i^r = \sum_{i=1}^5 b_i^r, \quad r = 1, 2, 3, 4,$$

where  $a_i = -b_i$ ,  $i = 1, 2, \dots, 5$ . The four equations of the system (7) now reduce to the following two equations:

$$(8) \quad a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

and

$$(9) \quad a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = 0.$$

Thus, to obtain the complete symmetric solution, in reduced form, of the diophantine system (7), we must obtain the complete solution in integers of the equations (8) and (9) such that  $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$ .

The equations (8) and (9) have trivial solutions in which one of the five integers is zero while the remaining four integers form two pairs, the sum of the integers in each pair being zero, as for example,  $(x_1, x_2, -x_1, -x_2, 0)$ .

Moreover, it is readily seen that if any solution of (8) and (9) is such that one of the five integers  $x_i$  is zero, or the sum of any two of the five integers  $x_i$  is zero, then the solution must be trivial. Further, trivial solutions of equations (8) and (9) lead to trivial symmetric solutions of (7).

We will now find the complete non-trivial solution of equations (8) and (9) such that  $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$ . Let  $x_i, i = 1, 2, \dots, 5$  be any such non-trivial solution of (8) and (9) so that  $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$  and the  $x_i$  satisfy the equations

$$(10) \quad x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$

and

$$(11) \quad x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

As our solution is assumed to be non-trivial, we must have  $x_1 \neq 0, x_2 \neq 0, (x_2 + x_3) \neq 0$  and  $(x_1 + x_4) \neq 0$  and, accordingly, there must exist non-zero integers  $p, q, r$  and  $s$  such that

$$(12) \quad px_1 = q(x_2 + x_3),$$

and

$$(13) \quad rx_2 = s(x_1 + x_4).$$

Solving the linear equations (10), (12) and (13), we get

$$(14) \quad \begin{aligned} x_3 &= (px_1 - qx_2)/q, \\ x_4 &= (rx_2 - sx_1)/s, \\ x_5 &= -(psx_1 + qrx_2)/(qs). \end{aligned}$$

Substituting these values of  $x_3, x_4$  and  $x_5$  in equation (11), we get, on simplification,

$$(15) \quad -3x_1x_2 \left[ \{(p^2s(r+s) - q^2rs)\}x_1 + \{pq(r^2 - s^2) + q^2r^2\}x_2 \right] / (q^2s^2) = 0.$$

As  $x_1x_2 \neq 0$ , it follows from (15) that

$$(16) \quad \begin{aligned} x_1 &= \rho^{-1} \{pq(r^2 - s^2) + q^2r^2\}, \\ x_2 &= -\rho^{-1} \{(p^2s(r+s) - q^2rs)\}, \end{aligned}$$

where  $\rho$  is some rational number. Substituting these values of  $x_1, x_2$  in (14), we get

$$(17) \quad \begin{aligned} x_3 &= \rho^{-1} \{p^2r(r+s) + pqr^2 - q^2rs\}, \\ x_4 &= -\rho^{-1} \{p^2r(r+s) + pq(r^2 - s^2)\}, \\ x_5 &= \rho^{-1} \{p^2s(r+s) - pqr^2 - q^2r^2\}. \end{aligned}$$

Thus, a given non-trivial solution  $x_i$ ,  $i = 1, 2, \dots, 5$  of equations (8) and (9) must be of the type given by (16) and (17) where  $p, q, r$  and  $s$  are certain integers and, as we assumed  $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$ , the rational number  $\rho$  must be an integer such that it ensures that  $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$ .

In accordance with the pattern of equations (16) and (17), we now write

$$(18) \quad \begin{aligned} a_1 &= \rho^{-1} \{pq(r^2 - s^2) + q^2r^2\}, \\ a_2 &= -\rho^{-1} \{(p^2s(r + s) - q^2rs)\}, \\ a_3 &= \rho^{-1} \{p^2r(r + s) + pqr^2 - q^2rs\}, \\ a_4 &= -\rho^{-1} \{p^2r(r + s) + pq(r^2 - s^2)\}, \\ a_5 &= \rho^{-1} \{p^2s(r + s) - pqr^2 - q^2r^2\}, \end{aligned}$$

where  $p, q, r$  and  $s$  are arbitrary integers and  $\rho$  is an integer so chosen that  $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$ . It is now readily verified by direct substitution that  $a_1, a_2, a_3, a_4, a_5$  as defined by (18) satisfy both the equations (8) and (9). It has already been seen that any given non-trivial solution of (8) and (9) is of the type (18), and hence it follows that this is the complete non-trivial solution of equations (8) and (9).

It now follows that the complete ideal symmetric solution of the Tarry-Escott problem of degree four is given in the reduced form by  $a_i = -b_i$ ,  $i = 1, 2, \dots, 5$ , where  $a_1, a_2, a_3, a_4, a_5$  are defined by (18) in terms of the arbitrary integer parameters  $p, q, r$  and  $s$  while  $\rho$  is an integer so chosen that  $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$ . Symmetric ideal solutions that are not in the reduced form may be obtained by the application of Frolov's theorem to the above symmetric ideal solution.

As a numerical example, when  $p = 1$ ,  $q = 1$ ,  $r = 2$ ,  $s = 1$ ,  $\rho = 1$ , we get, after suitable re-arrangement, the following reduced ideal symmetric solution of the Tarry-Escott problem of degree 4:

$$(-9)^r + (-5)^r + (-1)^r + 7^r + 8^r = (-8)^r + (-7)^r + 1^r + 5^r + 9^r, \quad r = 1, 2, 3, 4.$$

Adding the constant 10 to all the terms, we get the following symmetric solution in positive integers:

$$1^r + 5^r + 9^r + 17^r + 18^r = 2^r + 3^r + 11^r + 15^r + 19^r, \quad r = 1, 2, 3, 4.$$