IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE FOUR AND A RELATED DIOPHANTINE SYSTEM

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IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE FOUR AND A RELATED DIOPHANTINE SYSTEM

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ABSTRACT. In this paper, the complete ideal symmetric solution in integers of the Tarry-Escott problem of degree four, that is, of the system of simultaneous equations $\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r$, $r = 1,2,3,4$, has been obtained. In addition, a parametric ideal non-symmetric solution has also been obtained, and this non-symmetric solution has been used to obtain a parametric solution of the diophantine system $\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r$, $r = 1,2,3,4$ and 6.

1. INTRODUCTION

The Tarry-Escott problem of degree k consists of finding two sets of integers $a_1, a_2, \dots, a_s$ and $b_1, b_2, \dots, b_s$ such that

$$\sum_{i=1}^{s} a_i^r = \sum_{i=1}^{s} b_i^r, \qquad r = 1,2,\dots,k. \tag{1}$$

There is a well-known theorem [6, p. 614] due to Frolov according to which the relations (1) imply that

$$\sum_{i=1}^{s} (Ma_i + K)^r = \sum_{i=1}^{s} (Mb_i + K)^r, \qquad r = 1,2,\dots,k, \tag{2}$$

where M and K are arbitrary integers. That is, if $(a_1, a_2, \dots, a_s; b_1, b_2, \dots, b_s)$ is a solution of the system (1), then

$$(Ma_1 + K, \dots, Ma_s + K; Mb_1 + K, \dots, Mb_s + K)$$

is also a solution of (1). This theorem is easily established by using the binomial theorem. If one solution of the system (1) is obtained from another through the application of this theorem, the two are called equivalent solutions. When we speak of distinct solutions, we refer to solutions that are not equivalent.

It follows from Frolov's theorem that for each solution of (1), there is an equivalent one such that $\sum_{i=1}^{s} a_i = 0 = \sum_{i=1}^{s} b_i$ and the greatest common divisor of all the integers $a_1, a_2, \ldots, a_s$ and $b_1, b_2, \ldots, b_s$ is 1, that is, $\gcd(a_i, b_i) = 1$. This is said to be the *reduced form* of the solution.

It is easily established [6, p. 616] that for a non-trivial solution of (1) to exist, we must have $s \geq (k+1)$. Solutions of the system of equations (1) are called ideal if $s = (k+1)$ and are of particular interest in several applications [6].

In order to reduce the number of equations of the system (1), the following simplifying conditions are often imposed:

$$a_i = -b_i, \quad i = 1, 2, \ldots, s, \quad \text{for } s \text{ odd}, \tag{3}$$

or

$$a_{s+1-i} = -a_i, \quad b_{s+1-i} = -b_i, \quad i = 1, 2, \ldots, s/2, \quad \text{for } s \text{ even}. \tag{4}$$

Solutions of (1) subject to the conditions (3) or (4) are called symmetric solutions. The conditions of symmetry, together with the condition $\gcd(a_i, b_i) = 1$, ensure that such solutions are in reduced form. Solutions of (1) obtained by the application of Frolov's theorem to a symmetric solution are also considered symmetric as they are equivalent to a symmetric solution. Solutions of (1) that are not symmetric are called non-symmetric.

The complete ideal solution (whether symmetric or non-symmetric) of the Tarry-Escott problem of degrees 2 and 3 has been given by Dickson [4, pp. 52, 55–58] but for higher degrees the complete ideal solution is not known. When $4 \leq k \leq 7$, parametric ideal solutions of (1) are known but these are all symmetric [2; 3, pp. 304–305; 5; 7, pp. 41–54]. However, these parametric solutions do not even give the complete ideal symmetric solution for any $k \geq 4$. Numerical ideal symmetric solutions of (1) have been given by Letac [8] for $k = 8$, by Letac as well as Smyth [10] for $k = 9$ and recently a numerical ideal symmetric solution for $k = 11$ was discovered through the combined efforts of Nuutti Kuosa, Jean-Charles Meyrignac and Chen Shuwen [9]. Parametric ideal non-symmetric solutions of (1) are not known for any $k \geq 4$. A numerical ideal non-symmetric solution has been given by Gloden [7, p. 25] when $k = 4$. Moreover, the aforementioned numerical ideal solutions for $k = 9$ given by Letac and Smyth provide non-symmetric ideal solutions for $k = 4$ [1, p. 10].

It is interesting to observe that ideal non-symmetric solutions of (1) can be used to generate solutions of the system of equations

$$(5)\qquad \sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \qquad r = 1, 2, \ldots, k, k+2.$$

This follows from a theorem given by Gloden [7, p. 24]. Symmetric ideal solutions cannot be used effectively for this purpose as the solutions obtained by applying this theorem hold trivially either for all odd or for all even values of r according as k is odd or even.

In this paper, we will obtain the complete ideal symmetric solution of the Tarry-Escott problem of degree four as well as a parametric ideal non-symmetric solution of this problem. We shall use the non-symmetric solution to obtain a parametric solution of the system of equations

$$(6)\qquad \sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4, 6.$$

Parametric solutions of the system of equations (6) have not been obtained earlier.

2. The complete ideal symmetric solution of the Tarry-Escott problem of degree four

To obtain the complete ideal symmetric solution of degree four, we have to obtain a solution of the system of equations

$$(7)\qquad \sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4,$$

where $a_i = -b_i$, $i = 1, 2, \ldots, 5$. The four equations of the system (7) now reduce to the following two equations:

$$(8)\qquad a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

and

$$(9)\qquad a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = 0\,.$$

Thus, to obtain the complete symmetric solution, in reduced form, of the diophantine system (7), we must obtain the complete solution in integers of the equations (8) and (9) such that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$.

The equations (8) and (9) have trivial solutions in which one of the five integers is zero while the remaining four integers form two pairs, the sum of the integers in each pair being zero, as for example, $(x_1, x_2, -x_1, -x_2, 0)$.

Moreover, it is readily seen that if any solution of (8) and (9) is such that one of the five integers x_i is zero, or the sum of any two of the five integers x_i is zero, then the solution must be trivial. Further, trivial solutions of equations (8) and (9) lead to trivial symmetric solutions of (7).

We will now find the complete non-trivial solution of equations (8) and (9) such that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$. Let x_i, $i = 1, 2, \ldots, 5$ be any such non-trivial solution of (8) and (9) so that $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$ and the x_i satisfy the equations

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0, \tag{10}$$

and

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0. \tag{11}$$

As our solution is assumed to be non-trivial, we must have $x_1 \neq 0$, $x_2 \neq 0$, $(x_2 + x_3) \neq 0$ and $(x_1 + x_4) \neq 0$ and, accordingly, there must exist non-zero integers p, q, r and s such that

$$px_1 = q(x_2 + x_3), \tag{12}$$

and

$$rx_2 = s(x_1 + x_4). \tag{13}$$

Solving the linear equations (10), (12) and (13), we get

$$\begin{aligned} x_3 &= (px_1 - qx_2)/q, \\ x_4 &= (rx_2 - sx_1)/s, \\ x_5 &= -(psx_1 + qrx_2)/(qs). \end{aligned} \tag{14}$$

Substituting these values of x_3, x_4 and x_5 in equation (11), we get, on simplification,

$$-3x_1x_2\big[\{(p^2s(r+s) - q^2rs)\}x_1 + \{pq(r^2 - s^2) + q^2r^2\}x_2\big]/(q^2s^2) = 0. \tag{15}$$

As $x_1x_2 \neq 0$, it follows from (15) that

$$\begin{aligned} x_1 &= \rho^{-1}\{pq(r^2 - s^2) + q^2r^2\}, \\ x_2 &= -\rho^{-1}\{(p^2s(r+s) - q^2rs)\}, \end{aligned} \tag{16}$$

where ρ is some rational number. Substituting these values of x_1, x_2 in (14), we get

$$\begin{aligned} x_3 &= \rho^{-1}\{p^2r(r+s) + pqr^2 - q^2rs\}, \\ x_4 &= -\rho^{-1}\{p^2r(r+s) + pq(r^2 - s^2)\}, \\ x_5 &= \rho^{-1}\{p^2s(r+s) - pqr^2 - q^2r^2\}. \end{aligned} \tag{17}$$

Thus, a given non-trivial solution x_i, $i = 1, 2, \ldots, 5$ of equations (8) and (9) must be of the type given by (16) and (17) where p, q, r and s are certain integers and, as we assumed $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$, the rational number ρ must be an integer such that it ensures that $\gcd(x_1, x_2, x_3, x_4, x_5) = 1$.

In accordance with the pattern of equations (16) and (17), we now write

$$\begin{aligned} a_1 &= \rho^{-1}\{pq(r^2 - s^2) + q^2r^2\}, \\ a_2 &= -\rho^{-1}\{(p^2s(r+s) - q^2rs)\}, \\ a_3 &= \rho^{-1}\{p^2r(r+s) + pqr^2 - q^2rs\}, \\ a_4 &= -\rho^{-1}\{p^2r(r+s) + pq(r^2 - s^2)\}, \\ a_5 &= \rho^{-1}\{p^2s(r+s) - pqr^2 - q^2r^2\}, \end{aligned} \tag{18}$$

where p, q, r and s are arbitrary integers and ρ is an integer so chosen that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$. It is now readily verified by direct substitution that a_1, a_2, a_3, a_4, a_5 as defined by (18) satisfy both the equations (8) and (9). It has already been seen that any given non-trivial solution of (8) and (9) is of the type (18), and hence it follows that this is the complete non-trivial solution of equations (8) and (9).

It now follows that the complete ideal symmetric solution of the Tarry-Escott problem of degree four is given in the reduced form by $a_i = -b_i$, $i = 1, 2, \ldots, 5$, where a_1, a_2, a_3, a_4, a_5 are defined by (18) in terms of the arbitrary integer parameters p, q, r and s while ρ is an integer so chosen that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$. Symmetric ideal solutions that are not in the reduced form may be obtained by the application of Frolov's theorem to the above symmetric ideal solution.

As a numerical example, when $p = 1$, $q = 1$, $r = 2$, $s = 1$, $\rho = 1$, we get, after suitable re-arrangement, the following reduced ideal symmetric solution of the Tarry-Escott problem of degree 4:

$$(-9)^r + (-5)^r + (-1)^r + 7^r + 8^r = (-8)^r + (-7)^r + 1^r + 5^r + 9^r, \quad r = 1, 2, 3, 4.$$

Adding the constant 10 to all the terms, we get the following symmetric solution in positive integers:

$$1^r + 5^r + 9^r + 17^r + 18^r = 2^r + 3^r + 11^r + 15^r + 19^r, \quad r = 1, 2, 3, 4.$$

3. A PARAMETRIC IDEAL NON-SYMMETRIC SOLUTION OF THE TARRY-ESCOTT PROBLEM OF DEGREE FOUR

We will now solve the equations

(19) $$a_1 + a_2 + a_3 + a_4 + a_5 = b_1 + b_2 + b_3 + b_4 + b_5\,,$$

(20) $$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = b_1^2 + b_2^2 + b_3^3 + b_4^2 + b_5^2\,,$$

(21) $$a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = b_1^3 + b_2^3 + b_3^3 + b_4^3 + b_5^3\,,$$

(22) $$a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 = b_1^4 + b_2^4 + b_3^4 + b_4^4 + b_5^4\,,$$

so as to get an ideal non-symmetric solution of the Tarry-Escott problem of degree four. We write

(23) $$\begin{aligned} a_1 &= 2px - (\xi + \eta)y\,, & b_1 &= 2px + \eta y,\\ a_2 &= 2qx + \eta y\,, & b_2 &= 2qx - (\xi + \eta)y\,,\\ a_3 &= rx\,, & b_3 &= rx + \zeta y\,,\\ a_4 &= sx - \zeta y\,, & b_4 &= sx\,,\\ a_5 &= \zeta y\,, & b_5 &= -\zeta y\,. \end{aligned}$$

We will first choose p, q, r, s, ξ, η and ζ such that equations (19), (20) and (21) are identically satisfied for all values of x and y. In the equation obtained from (22) by substituting the values of a_i, b_i as given above, the coefficients of x^4 and y^4 on the two sides are equal, and we will choose p, q, r, s, ξ, η and ζ so as to satisfy the additional condition that the coefficient of xy^3 also becomes equal on both sides of this equation. Thus, equation (22) would reduce to an equation containing only the terms x^3y and x^2y^2 and accordingly it can be readily solved for x and y. These values of x and y together with the already suitably chosen values of p, q, r, s, ξ, η and ζ substituted in (23) will give a solution of equations (19), (20), (21) and (22).

When a_i, b_i are defined by (23), we observe that equation (19) is identically satisfied. Substituting the values of a_i, b_i in (20), we note that this equation will also be identically satisfied for all values of x and y if the following condition is satisfied:

(24) $$2(\xi + 2\eta)(p - q) + \zeta(r + s) = 0\,.$$

Next, we substitute the values of a_i, b_i as given by (23) in equation (21) and observe that the coefficients of x^3 and y^3 on both sides are equal. Equating the coefficients of x^2y and xy^2 on both sides of this equation, we get the following two conditions:

(25) $$4(\xi + 2\eta)(p^2 - q^2) + \zeta(r^2 + s^2) = 0\,,$$

and

$$2\xi(\xi+2\eta)(p-q)-\zeta^2 r+\zeta^2 s=0\,. \tag{26}$$

Finally, in the equation obtained by substituting the values of a_i, b_i in (22), we equate, as already discussed, the coefficients of xy^3 on both sides to get the additional condition:

$$2(\xi^3+3\xi^2\eta+3\xi\eta^2+2\eta^3)(p-q)+\zeta^3(r+s)=0\,. \tag{27}$$

We now proceed to solve equations (24), (25), (26) and (27). Equations (24) and (27) may be considered as two linear equations in the two linear variables $(p-q)$ and $(r+s)$, and they will be consistent only if ξ, η and ζ satisfy the condition

$$(\xi+2\eta)(\xi^2+\xi\eta+\eta^2-\zeta^2)=0\,.$$

Taking $(\xi + 2\eta) = 0$ leads to trivial solutions, so we will choose ξ, η and ζ such that

$$\xi^2+\xi\eta+\eta^2-\zeta^2=0\,. \tag{28}$$

The complete solution of (28) is readily found to be

$$\xi=2mn-m^2,\qquad \eta=m^2-n^2,\qquad \zeta=m^2-mn+n^2\,. \tag{29}$$

Next, we solve equations (24) and (26) for r and s, and substitute their values in equation (25) which now has a linear factor $(p-q)$ that can be ignored and then equation (25) is readily seen to be satisfied if we choose p and q as follows:

$$\begin{aligned} p&=\xi^3+2\xi^2\eta+\xi\zeta^2+2\eta\zeta^2-2\zeta^3\,,\\ q&=\xi^3+2\xi^2\eta+\xi\zeta^2+2\eta\zeta^2+2\zeta^3\,. \end{aligned} \tag{30}$$

With these values of p and q, we immediately get

$$\begin{aligned} r&=-4\xi^2\zeta-8\xi\eta\zeta+4\xi\zeta^2+8\eta\zeta^2\,,\\ s&=4\xi^2\zeta+8\xi\eta\zeta+4\xi\zeta^2+8\eta\zeta^2\,. \end{aligned} \tag{31}$$

Thus, when ξ, η, ζ are defined by (29), and p, q, r and s are given by (30) and (31), equations (24), (25), (26) and (27) are all satisfied. With these values of p, q, r, s, ξ, η and ζ, equation (22) reduces, on removing the factor $64x^2y\zeta^3(\xi+2\eta)$, to

$$\begin{aligned}(6\xi^6+24\xi^5\eta+24\xi^4\eta^2-12\xi^4\zeta^2-48\xi^3\eta\zeta^2-48\xi^2\eta^2\zeta^2-2\xi^2\zeta^4\\ -8\xi\eta\zeta^4-8\eta^2\zeta^4+8\zeta^6)x-(3\xi^4+6\xi^3\eta-3\xi^2\zeta^2-6\xi\eta\zeta^2)y=0\,.\end{aligned}$$

Thus, equation (22) will be satisfied if we choose

$$\begin{aligned} x &= 3\xi^4 + 6\xi^3\eta - 3\xi^2\zeta^2 - 6\xi\eta\zeta^2, \\ y &= 6\xi^6 + 24\xi^5\eta + 24\xi^4\eta^2 - 12\xi^4\zeta^2 - 48\xi^3\eta\zeta^2 \\ &\quad - 48\xi^2\eta^2\zeta^2 - 2\xi^2\zeta^4 - 8\xi\eta\zeta^4 - 8\eta^2\zeta^4 + 8\zeta^6. \end{aligned} \tag{32}$$

A solution of equations (19), (20), (21) and (22) can now be obtained in terms of the parameters m and n by taking ξ, η and ζ as given by (29), substituting these values of ξ, η and ζ in (30), (31) and (32) to obtain p, q, r, s, x and y in terms of m and n, and then substituting the values of p, q, r, s, ξ, η, ζ, x and y in (23). The solution so obtained may, after simplification and removal of common factors, be written explicitly in terms of the arbitrary parameters m and n as follows:

$$\begin{aligned} a_1 &= 12m^7n - 37m^6n^2 + 24m^5n^3 + 12m^4n^4 - 20m^3n^5 \\ &\quad + 15m^2n^6 - 18mn^7 + 8n^8, \\ a_2 &= 10m^7n - 30m^6n^2 + 54m^5n^3 - 13m^4n^4 - 48m^3n^5 \\ &\quad + 45m^2n^6 - 14mn^7, \\ a_3 &= 4m^8 + 6m^7n - 28m^6n^2 + 8m^5n^3 + 66m^4n^4 \\ &\quad - 128m^3n^5 + 112m^2n^6 - 48mn^7 + 8n^8, \\ a_4 &= 4m^8 - 12m^7n + 35m^6n^2 - 55m^5n^3 + 66m^4n^4 \\ &\quad - 65m^3n^5 + 49m^2n^6 - 30mn^7 + 8n^8, \\ a_5 &= -4m^8 + 14m^7n - 27m^6n^2 + 55m^5n^3 - 80m^4n^4 \\ &\quad + 81m^3n^5 - 49m^2n^6 + 14mn^7, \\ b_1 &= -4m^8 + 14m^7n - 22m^6n^2 + 10m^5n^3 + 67m^4n^4 \\ &\quad - 140m^3n^5 + 113m^2n^6 - 46mn^7 + 8n^8, \\ b_2 &= 4m^8 + 8m^7n - 45m^6n^2 + 68m^5n^3 - 68m^4n^4 \\ &\quad + 72m^3n^5 - 53m^2n^6 + 14mn^7, \\ b_3 &= 20m^7n - 55m^6n^2 + 63m^5n^3 - 14m^4n^4 - 47m^3n^5 \\ &\quad + 63m^2n^6 - 34mn^7 + 8n^8, \\ b_4 &= 2m^7n + 8m^6n^2 - 14m^4n^4 + 16m^3n^5 - 16mn^7 + 8n^8, \\ b_5 &= 4m^8 - 14m^7n + 27m^6n^2 - 55m^5n^3 + 80m^4n^4 \\ &\quad - 81m^3n^5 + 49m^2n^6 - 14mn^7. \end{aligned} \tag{33}$$

We may apply Frolov's theorem to the above solution to obtain other non-symmetric solutions. For instance, an arbitrary constant K can be added to all the terms a_i, b_i, $i = 1, 2, 3, 4, 5$.

As a numerical example, taking $m = 3$, $n = 1$, we get, on suitable re-arrangement, the following solution:

$$(-1659)^r + 1406^r + 2784^r + 4025^r + 5915^r$$
$$= (-1675)^r + 1659^r + 2366^r + 4256^r + 5865^r,$$

where $r = 1, 2, 3, 4$. Adding the constant 1676 to all the terms, we get the following solution in positive integers:

$$17^r + 3082^r + 4460^r + 5701^r + 7591^r = 1^r + 3335^r + 4042^r + 5932^r + 7541^r,$$

where $r = 1, 2, 3, 4$.

4. THE DIOPHANTINE SYSTEM $\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r$, $r = 1, 2, 3, 4, 6$

We will now state the theorem given by Gloden [7, p. 24] to which a reference has already been made in the introduction and then apply it to obtain a parametric solution of this diophantine system.

THEOREM 4.1. *If*

$$\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \qquad r = 1, 2, \ldots, k$$

then

$$\sum_{i=1}^{k+1} (a_i + t)^r = \sum_{i=1}^{k+1} (b_i + t)^r, \qquad r = 1, 2, \ldots, k, k+2,$$

where

$$t = -\left(\sum_{i=1}^{k+1} a_i\right)/(k+1).$$

As we have already obtained, in the preceding section, a parametric solution of $\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r$, $r = 1, 2, 3, 4$, a direct application of the above theorem gives a parametric solution of $\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r$, $r = 1, 2, 3, 4$ and 6. We multiply the $(a_i + t)$, $(b_i + t)$, $i = 1, 2, 3, 4, 5$ by 5 to cancel out denominators, and we rename the resulting expressions as a_i, b_i, $i = 1, 2, 3, 4, 5$, so that the parametric solution of the diophantine system

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4, 6$$

may be written as

$$\begin{aligned}
a_1 &= 4m^8 - 30m^7n + 98m^6n^2 - 34m^5n^3 - 9m^4n^4 \\
&\quad - 80m^3n^5 + 97m^2n^6 - 6mn^7 - 16n^8, \\
a_2 &= 4m^8 - 20m^7n + 63m^6n^2 - 184m^5n^3 + 116m^4n^4 \\
&\quad + 60m^3n^5 - 53m^2n^6 - 26mn^7 + 24n^8, \\
a_3 &= -16m^8 + 53m^6n^2 + 46m^5n^3 - 279m^4n^4 \\
&\quad + 460m^3n^5 - 388m^2n^6 + 144mn^7 - 16n^8, \\
a_4 &= -16m^8 + 90m^7n - 262m^6n^2 + 361m^5n^3 - 279m^4n^4 \\
&\quad + 145m^3n^5 - 73m^2n^6 + 54mn^7 - 16n^8, \\
a_5 &= 24m^8 - 40m^7n + 48m^6n^2 - 189m^5n^3 + 451m^4n^4 \\
&\quad - 585m^3n^5 + 417m^2n^6 - 166mn^7 + 24n^8, \\
b_1 &= 24m^8 - 40m^7n + 23m^6n^2 + 36m^5n^3 - 284m^4n^4 \\
&\quad + 520m^3n^5 - 393m^2n^6 + 134mn^7 - 16n^8, \\
b_2 &= -16m^8 - 10m^7n + 138m^6n^2 - 254m^5n^3 + 391m^4n^4 \\
&\quad - 540m^3n^5 + 437m^2n^6 - 166mn^7 + 24n^8, \\
b_3 &= 4m^8 - 70m^7n + 188m^6n^2 - 229m^5n^3 + 121m^4n^4 \\
&\quad + 55m^3n^5 - 143m^2n^6 + 74mn^7 - 16n^8, \\
b_4 &= 4m^8 + 20m^7n - 127m^6n^2 + 86m^5n^3 + 121m^4n^4 \\
&\quad - 260m^3n^5 + 172m^2n^6 - 16mn^7 - 16n^8, \\
b_5 &= -16m^8 + 100m^7n - 222m^6n^2 + 361m^5n^3 - 349m^4n^4 \\
&\quad + 225m^3n^5 - 73m^2n^6 - 26mn^7 + 24n^8.
\end{aligned}$$

As a numerical example, when $m = 3$ and $n = 1$, we get, on removal of common factors and suitable re-arrangement, the following solution:

$$\begin{aligned}
&1449^r + 7654^r + 17104^r + (-5441)^r + (-20766)^r \\
&\qquad = 8809^r + 16854^r + (-641)^r + (-4176)^r + (-20846)^r,
\end{aligned}$$

where $r = 1, 2, 3, 4$ and 6.

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