1. Orderable groups

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contrast, there are uncountably many infinite type orderings, and all of them are non-discrete.

The outline of the paper is as follows. In the first section we give a short introduction to orderable groups and survey some known results about them. In the second section we present Thurston's construction. In the third section we define finite and infinite type orderings, and state our classification theorems. Sections four to six are concerned with finite type orderings: in Section 4 we describe a different method of constructing orderings, using "curve diagrams". In Section 5 we prove that the set of orderings arising from curve diagrams is very easy to understand and classify. Moreover, we prove that up to conjugacy only a finite number of orderings arise in this way. In the sixth section we prove the classification theorems for finite type orderings. The strategy is to associate to every point of $\mathbf{R}$ with orbit of finite type a curve diagram such that the orderings arising from this point and from the curve diagram agree. Thus we obtain, via curve diagram orderings, a good understanding of Thurston type orderings. In Section 7 we prove the results about the infinite type case.

1. Orderable groups

In this section we define orderable groups and survey some known results about them. The standard reference for orderings on groups is Rhemtulla and Mura's book [19].

DEFINITION 1.1. A group G is *left orderable* (respectively *right orderable*) if there is a total order $<$ on G which is invariant under left mutiplication (resp. right multiplication), that is, such that, for all $a, b \in G$, $a < b$, $a = b$ or $b < a$, and for all $g \in G$, $a < b$ implies that $ga < gb$ (resp. $ag < bg$).

A group G is *bi-orderable* or *two-sided orderable* if there is a total order on G which is respected by multiplication on the left and multiplication on the right: i.e. $a < b \Longrightarrow ga < gb$ and $ag < bg$.

Two left orderings $<$ and $\prec$ on a group G are *conjugate* if there exists a $g \in G$ such that $a \prec b$ if and only if $ag < bg$. So two left orderings are conjugate if "one is obtained from the other by right translation in the group".

REMARKS 1.2.

(1) The following observation will be extremely important in what follows. If a group G acts on the left by orientation preserving homeomorphisms on $\mathbf{R}$, then every point x in $\mathbf{R}$ with free orbit (i.e. $Stab(x) = \{1_G\}$) gives rise to a left ordering on G, by defining $g > h \;:\Longleftrightarrow\; g(x) > h(x)$. We have for every $f \in G$ that $fg(x) > fh(x) \iff g(x) > h(x)$, since the action of f preserves the orientation of $\mathbf{R}$; this implies that the ordering is indeed left invariant. Note that different points in $\mathbf{R}$ may give rise to different orderings. If a point x does not have free orbit, it still gives rise to a *partial* left invariant ordering.

(2) In fact, a countable group is left orderable if and only if it has an action by orientation preserving homeomorphisms on $\mathbf{R}$ such that only the trivial group element acts by the identity-homeomorphism, see for instance [11].

(3) A left orderable group is torsion-free: if an element x had order n, and if $1 < x$, then it would follow that $1 < x < x^2 < \cdots < x^{n-1} < x^n = 1$.

(4) The "positive cone" of the ordering, $P = \{g \in G \mid g > 1\}$ has the properties that $G = P \sqcup \{1\} \sqcup P^{-1}$, and that $PP \subset P$. Conversely, given a subset with these two properties, a left order $<$ can be defined by $a < b \;:\Longleftrightarrow\; a^{-1}b \in P$. Similarly, a right order $\prec$ is obtained from $a \prec b \;:\Longleftrightarrow\; ab^{-1} \in P$. (In particular, a group is left orderable if and only if it is right orderable.) The orders are total because of the first property, and transitive because of the second. The orders are bi-orders if and only if we have in addition that $g^{-1}Pg \subseteq P$ for all $g \in G$.

(5) The following classes of groups are bi-orderable:

(a) finitely generated torsion-free abelian groups;

(b) finitely generated free groups (this is a result of Magnus, see e.g. [13]);

(c) more generally, residually free groups, like fundamental groups of closed surfaces (this is due to Baumslag, see [26, 27, 28]).

(6) If S is a *closed* surface, then $\mathcal{MCG}(S)$ has torsion, but there exists a finite index subgroup which is torsion-free: consider the set of all elements which act as the identity on the homology $H_1(S, \mathbf{Z}_p)$, where p is a prime larger than $84(\text{genus} - 1)$. The torsion-freeness of these groups seems to be a folklore result, the analogue for the Torelli group (defined in the same way, only with $\mathbf{Z}_p$ replaced by $\mathbf{Z}$) is proved in [14]. It is an open problem whether or not these subgroups are left orderable.

We now give four examples of attractive results about orders on groups.

(1) Neville Smythe [23] used the orderability of surface groups to prove that any null-homotopic curve on a surface S is the image under projection of an embedded unknotted loop in $S \times I$.

(2) As pointed out by N. Smythe [16] in response to a question of L. Neuwirth [15, Question N], knot groups are in general not bi-orderable. For instance the trefoil knot group (which is isomorphic to the braid group on three strings B_3), is not bi-orderable. To show this, recall that B_3 contains an element Δ (the "half twist") which is not in the centre, but whose square Δ^2 is. Assume that $>$ is a bi-ordering of B_3, and let $b \in B_3$ be such that $b\Delta \neq \Delta b$, say $b\Delta > \Delta b$. Multiplying this inequality on the left by Δ and on the right by Δ^{-1} would yield $\Delta b > \Delta^2 b \Delta^{-1} = b\Delta^2\Delta^{-1} = b\Delta$, which is a contradiction.

Neuwirth reformulated the question as 'Are knot groups left orderable ?'. A positive answer to this question follows from an observation by J. Howie and H. Short [12] that knot groups are locally indicable (every non-trivial finitely generated subgroup has $\mathbf{Z}$ as a homomorphic image), together with a theorem of Burns and Hale [4] that locally indicable groups are left orderable. The converse of Burns and Hale's theorem is known to be false – see [1] and [9, Theorem 5.3].

(3) We have just seen that B_3 (and hence B_n for all n) is not bi-orderable. Kim and Rolfsen [13] have recently proved that the finite index subgroup PB_n of *pure* braids *is* bi-orderable. However, no bi-ordering of PB_n extends to a left ordering of B_n [20].

(4) The Zero Divisor Conjecture, often attributed to Kaplansky, asserts that if R is a ring without zero divisors and G is a torsion-free group then the group ring RG has no zero divisors. The hypothesis that G be torsion-free is necessary, for if G contains an element x of order n then $(1-x)(1+x+\cdots+x^{n-1}) = 0$ in RG. The conjecture is known to hold for left orderable groups. In fact, it is not hard to see that left orderable groups have the "two unique product" property which implies that the conjecture holds for them (see e.g. [18], and also Delzant [7] and Bowditch [3] for some recent remarks about this property).

2. Orderings of mapping class groups using hyperbolic geometry

In this section we present the construction of orders on mapping class groups of surfaces which we learned from W. P. Thurston, and prove that they