where the power series converges in the domain $\mathfrak{D}$, *and*

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases} \qquad \square$$

Since $L_p(s, \tau; \chi)$ is defined for each $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq 1$, we now have a p-adic function of two variables, $L_p(s, t; \chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$.

4. Properties of $L_p(s, t; \chi)$

Most of the properties that follow are direct consequences of similar properties that hold for the generalized Bernoulli polynomials. In all of the following we will take p prime and χ a Dirichlet character with conductor f_χ.

4.1 A symmetry property in t

The first property we obtain regarding $L_p(s, t; \chi)$ is a direct consequence of the generalized Bernoulli polynomials being either odd or even functions, except when $\chi = 1$. Recall that $L_p(s, t; \chi)$ interpolates the values

$$L_p(1 - n, t; \chi) = -\frac{1}{n} b_n(t) , \tag{18}$$

for $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, where

$$b_n(t) = B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1} qt\right) , \tag{19}$$

and we define

$$c_n(t) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m(t) . \tag{20}$$

Lemma 4.1. *For all* $n \in \mathbf{Z}$, $n \geq 0$, *we have*

$$B_{n,1}(-t) = (-1)^n B_{n,1}(t) - (-1)^n n t^{n-1} .$$

Proof. This holds for $n = 0$ since $B_{0,1}(t) = 1$. Now assume that $n \geq 1$. Because $B_{n,1} = 0$ for odd $n \geq 3$, we can write (2) in the form

$$B_{n,1}(t) = \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + n B_{1,1} t^{n-1} .$$

Any m such that $n-m$ is even must have the same parity as n. Thus

$$\begin{aligned} B_{n,1}(-t) &= (-1)^n \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + (-1)^{n-1} n B_{1,1} t^{n-1} \\ &= (-1)^n B_{n,1}(t) - 2(-1)^n n B_{1,1} t^{n-1}. \end{aligned}$$

From the value $B_{1,1} = -B_1 = 1/2$, the lemma then follows. □

LEMMA 4.2. *For all* $n \in \mathbf{Z}$, $n \geq 0$,

$$b_n(-t) = \chi(-1) b_n(t).$$

Proof. This is obviously true for $n = 0$ since

$$b_0(t) = \left(1 - \chi(p)p^{-1}\right) B_{0,\chi},$$

and $B_{0,\chi} = 0$ except when $\chi = 1$, in which case $B_{0,1} = 1$. So we can assume that $n \geq 1$.

First consider the case of $\chi_n = 1$. This implies that $\chi = \omega^n$. By Lemma 4.1,

$$\begin{aligned} b_n(-t) &= B_{n,1}(-qt) - p^{n-1} B_{n,1}\left(-p^{-1}qt\right) \\ &= (-1)^n B_{n,1}(qt) - (-1)^n n (qt)^{n-1} \\ &\quad - p^{n-1}\left((-1)^n B_{n,1}\left(p^{-1}qt\right) - (-1)^n n \left(p^{-1}qt\right)^{n-1}\right) \\ &= (-1)^n \left(B_{n,1}(qt) - p^{n-1} B_{n,1}\left(p^{-1}qt\right)\right) \\ &= (-1)^n b_n(t). \end{aligned}$$

Since $\chi = \omega^n$ and $\omega(-1) = -1$, the lemma holds for $\chi_n = 1$.

Now suppose that $\chi_n \neq 1$. Then, from (3),

$$\begin{aligned} b_n(-t) &= B_{n,\chi_n}(-qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(-p^{-1}qt\right) \\ &= (-1)^n \chi_n(-1)\left(B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1}qt\right)\right) \\ &= (-1)^n \chi_n(-1) b_n(t). \end{aligned}$$

Note that $\chi_n = \chi\omega^{-n}$, which implies that $\chi_n(-1) = (-1)^n \chi(-1)$. Thus the lemma also holds for $\chi_n \neq 1$.

Since the lemma holds for both $\chi_n = 1$ and $\chi_n \neq 1$, the proof must be complete. □

Using this result, we can prove

THEOREM 4.3. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s, -t; \chi) = \chi(-1) L_p(s, t; \chi).$$

Proof. From Lemma 4.2 we see that

$$b_n(-t) = \chi(-1) b_n(t).$$

Also, (20) implies that

$$c_n(-t) = \chi(-1) c_n(t).$$

From (16), whenever $n \geq -1$,

$$a_n(-t) = \chi(-1) a_n(t),$$

which implies that

$$L_p(s, -t; \chi) = \chi(-1) L_p(s, t; \chi). \qquad \square$$

If $\chi(-1) = -1$ and $t = 0$, then

$$L_p(s, 0; \chi) = -L_p(s, 0; \chi),$$

which implies that

$$L_p(s; \chi) = -L_p(s; \chi),$$

and thus $L_p(s; \chi) = 0$ for all $s \in \mathfrak{D}$, as we would expect.

4.2 $L_p(s, t; \chi)$ AS A POWER SERIES IN $t - \alpha$, $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$

To develop $L_p(s, t; \chi)$ in terms of a power series in t will enable us to find a derivative of this function with respect to this second variable. All this we shall do, but before doing so we need to specify some notation.

LEMMA 4.4. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$. *Then for* $n \in \mathbf{Z}$, $n \geq 1$,

$$\lim_{s \to 1-n} \binom{-s}{n} L_p(s + n, t; \chi) = -\frac{1}{n} \left(1 - \chi(p) p^{-1}\right) B_{0,\chi}.$$

Proof. Recall that, from Theorem 3.13, we can write

$$L_p(s, t; \chi) = \frac{a_{-1}(t)}{s - 1} + \sum_{m=0}^{\infty} a_m(t)(s - 1)^m,$$

where $a_{-1}(t) = (1 - \chi(p) p^{-1}) B_{0,\chi}$. Thus

$$\lim_{s\to 1}(s-1)L_p(s,t;\chi) = \left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.$$

Now let $n \in \mathbf{Z}$, $n \geq 1$, and consider

$$\lim_{s\to 1-n} \binom{-s}{n} L_p(s+n,t;\chi) = \lim_{s\to 1} \binom{n-s}{n} L_p(s,t;\chi)\,.$$

If $n=1$, then we write this as

$$\lim_{s\to 1}(1-s)L_p(s,t;\chi) = -\left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.$$

If $n \geq 2$, then

$$\frac{1}{n!}\lim_{s\to 1}\prod_{i=0}^{n-2}(n-s-i) = \frac{1}{n},$$

which implies that

$$\begin{aligned}\lim_{s\to 1-n} \binom{-s}{n} L_p(s+n,t;\chi) &= \frac{1}{n!}\left(\lim_{s\to 1}\prod_{i=0}^{n-2}(n-s-i)\right)\left(\lim_{s\to 1}(1-s)L_p(s,t;\chi)\right)\\ &= -\frac{1}{n}\left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.\end{aligned}$$

Therefore the lemma holds for all $n \geq 1$. □

Now, because $L_p(s,t;1)$ is undefined when $s=1$, the quantity

$$\binom{-s}{n} L_p(s+n,t;1)$$

is undefined when $s=1-n$, for $n \in \mathbf{Z}$, $n \geq 1$. However, Lemma 4.4 shows that this quantity exists as $s \to 1-n$. In the following we will encounter expressions that involve $\binom{-s}{n}L_p(s+n,t;\chi)$, and because of Lemma 4.4 we shall assume the understanding that

$$\left.\binom{-s}{n} L_p(s+n,t;\chi)\right|_{s=1-n} = -\frac{1}{n}\left(1-\chi(p)p^{-1}\right) B_{0,\chi}$$

for $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.5. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m t^m L_p(s+m;\chi_m)\,. \tag{21}$$

Proof. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $k \in \mathbf{Z}$, $k \geq 1$. Then

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p(1-k+m;\chi_m) = -\frac{1}{k}q^k t^k(1-\chi_k(p)p^{-1})B_{0,\chi_k} + \sum_{m=0}^{k-1}\binom{k-1}{m}q^m t^m L_p(1-(k-m);\chi_m).$$

By evaluating the L-function, we obtain

$$\binom{k-1}{m}L_p(1-(k-m);\chi_m) = -\frac{1}{k}\binom{k}{m}\left(1-\chi_k(p)p^{k-m-1}\right)B_{k-m,\chi_k},$$

and thus

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p\big(1-(k-m);\chi_m\big) = -\frac{1}{k}\sum_{m=0}^{k}\binom{k}{m}q^m t^m\big(1-\chi_k(p)p^{k-m-1}\big)B_{k-m,\chi_k},$$

which implies that the sum converges for $s=1-k$. Breaking this into two sums

$$\begin{aligned}
&\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p(1-(k-m);\chi_m)\\
&= -\frac{1}{k}\sum_{m=0}^{k}\binom{k}{m}B_{k-m,\chi_k}q^m t^m + \frac{1}{k}\chi_k(p)p^{k-1}\sum_{m=0}^{k}\binom{k}{m}B_{k-m,\chi_k}p^{-m}q^m t^m\\
&= -\frac{1}{k}\left(B_{k,\chi_k}(qt) - \chi_k(p)p^{k-1}B_{k,\chi_k}\left(p^{-1}qt\right)\right)\\
&= L_p(1-k,t;\chi)\ .
\end{aligned}$$

Thus (21) holds for a sequence $\{1-k\}_{k=1}^{\infty}$ that has 0 as a limit point. Lemma 2.5 then implies that Theorem 4.5 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (21).

Now we will show that the domains, in s, of each of the functions on either side of (21) contain $\mathfrak{D}$, except $s\neq 1$ when $\chi=1$.

This is obvious for the function $L_p(s,t;\chi)$. Consider the function

$$\sum_{m=0}^{\infty}\binom{-s}{m}q^m t^m L_p(s+m;\chi_m) = \sum_{m=0}^{\infty}\sum_{n=-1}^{\infty}\binom{-s}{m}q^m t^m a_{n,\chi_m}(s+m-1)^n.$$

We have seen that this sum converges for $s=1-k$, where $k\in\mathbf{Z}$, $k\geq 1$. Now we need to show that it converges for $s=\xi$, where $\xi\in\mathfrak{D}$, $\xi\neq 1$ if $\chi=1$, and $\xi\neq 1-k$ for $k\in\mathbf{Z}$, $k\geq 1$. So let ξ satisfy these restrictions, and let $\epsilon > 0$. Note that $|\xi - 1|_p < r$, where $r = |p|_p^{1/(p-1)}|q|_p^{-1}$. Let $r_0 \in \mathbf{R}$, $0 \le r_0 < r$, such that $|\xi - 1|_p = r_0$. Then for any $m \in \mathbf{Z}$, $m \ge 0$,

$$\begin{aligned} |\xi + m - 1|_p &\le \max\left\{|m|_p, |\xi - 1|_p\right\} \\ &\le \max\{1, r_0\}, \end{aligned}$$

implying that $\xi + m \in \mathfrak{D}$, $\xi + m \ne 1$. Let $\delta \in \mathbf{R}$ such that $r^\delta = \max\{1, r_0\}$. Then $0 \le \delta < 1$, and

$$|\xi + m - 1|_p \le r^\delta. \tag{22}$$

Let $N_1 \in \mathbf{Z}$ such that

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(N_1-1)/(p-1)} |q|_p^{(1-\delta)(N_1-1)} < \epsilon.$$

Then for any $m \in \mathbf{Z}$, $m \ge 1$, such that $m \ge N_1$, we must also have

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)} < \epsilon.$$

For $m \in \mathbf{Z}$, $m \ge 1$, consider

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \le |p|_p^{-1}|q|_p^m \left|\binom{-\xi}{m}(\xi + m - 1)^{-1}\right|_p.$$

Note that, by (22),

$$\begin{aligned} \left|\binom{-\xi}{m}(\xi + m - 1)^{-1}\right|_p &= |\xi + m - 1|_p^{-1} \prod_{i=1}^m \frac{|-\xi - (i-1)|_p}{|i|_p} \\ &\le |m!|_p^{-1} r^{\delta(m-1)}. \end{aligned}$$

Therefore

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \le |p|_p^{-1}|q|_p^m |m!|_p^{-1} r^{\delta(m-1)},$$

and from the bound

$$|m!|_p \ge |p|_p^{(m-1)/(p-1)},$$

we obtain

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p \le |p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)}.$$

Thus if $m \ge N_1$, then

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m}(\xi + m - 1)^{-1}\right|_p < \epsilon.$$

Now let $N_2 \in \mathbf{Z}$ such that

$$|f_\chi p|_p^{-1}|p|_p^{-(1-\delta)N_2/(p-1)}|q|_p^{(1-\delta)N_2} < \epsilon .$$

Then we must also have

$$|f_\chi p|_p^{-1}|p|_p^{-(1-\delta)(m+n)/(p-1)}|q|_p^{(1-\delta)(m+n)} < \epsilon$$

for any $m,n \in \mathbf{Z}$ such that $m \geq 0$, $n \geq 0$, and $\max\{m,n\} \geq N_2$. Let us consider

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m}(\xi+m-1)^n\right|_p \leq \left|\binom{-\xi}{m}\right|_p |q|_p^m |a_{n,\chi_m}|_p |\xi+m-1|_p^n ,$$

where $m,n \in \mathbf{Z}$, $m \geq 0$, $n \geq 0$. For all $m \geq 0$,

$$\left|\binom{-\xi}{m}\right|_p \leq |m!|_p^{-1} r^{\delta m},$$

and by utilizing this along with (17) and (22), our expression becomes

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m}(\xi+m-1)^n\right|_p \leq |m!(n+1)!|_p^{-1}|f_\chi p|_p^{-1} r^{\delta(m+n)}|q|_p^{m+n} .$$

Since

$$|m!(n+1)!|_p \geq |p|_p^{(m+n)/(p-1)},$$

we obtain

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m}(\xi+m-1)^n\right|_p \leq |f_\chi p|_p^{-1}|p|_p^{-(1-\delta)(m+n)/(p-1)}|q|_p^{(1-\delta)(m+n)} .$$

Thus if $\max\{m,n\} \geq N_2$, then

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m}(\xi+m-1)^n\right|_p < \epsilon .$$

Let $N = \max\{N_1,N_2\}$, and let $m,n \in \mathbf{Z}$, $m \geq 0$, $n \geq -1$. Then for $\max\{m,n\} \geq N$, it must be true that

$$\left|\binom{-\xi}{m} q^m t^m a_{n,\chi_m}(\xi+m-1)^n\right|_p < \epsilon .$$

Thus, by Proposition 2.4, the sum

$$\sum_{m=0}^{\infty}\sum_{n=-1}^{\infty}\binom{-\xi}{m} q^m t^m a_{n,\chi_m}(\xi+m-1)^n$$

must converge. This implies that the function on the right of (21) must converge for all $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$, and the theorem must then hold. □

Since we can now express $L_p(s,t;\chi)$ in terms of a power series in t, we can take a derivative of this function with respect to t.

LEMMA 4.6. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n! q^n \binom{-s}{n} L_p(s+n, t; \chi_n),$$

for $n \in \mathbf{Z}$, $n \geq 0$.

Proof. If $n = 0$, then the lemma is obviously true. So consider $n = 1$. Applying Proposition 2.6 to (21),

$$\frac{\partial}{\partial t} L_p(s, t; \chi) = \sum_{m=1}^{\infty} \binom{-s}{m} q^m m t^{m-1} L_p(s+m; \chi_m).$$

Now,

$$m \binom{-s}{m} = -s \binom{-s-1}{m-1},$$

so that

$$\begin{aligned}
\frac{\partial}{\partial t} L_p(s, t; \chi) &= \sum_{m=1}^{\infty} (-s) \binom{-s-1}{m-1} q^m t^{m-1} L_p(s+m; \chi_m) \\
&= -qs \sum_{m=0}^{\infty} \binom{-s-1}{m} q^m t^m L_p\left(s+1+m; \chi_{1+m}\right) \\
&= -qs L_p(s+1, t; \chi_1).
\end{aligned}$$

Now suppose that

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n! q^n \binom{-s}{n} L_p(s+n, t; \chi_n)$$

for some $n \in \mathbf{Z}$, $n \geq 1$. Then

$$\begin{aligned}
\frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) &= \frac{\partial}{\partial t}\left(\frac{\partial^n}{\partial t^n} L_p(s, t; \chi)\right) \\
&= n! q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s+n, t; \chi_n).
\end{aligned}$$

From the case for $n = 1$, we see that

$$\begin{aligned}
n! q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s+n, t; \chi_n) &= n! q^n \binom{-s}{n} (-s-n) q L_p\left(s+n+1, t; \chi_{n+1}\right) \\
&= (n+1)! q^{n+1} \binom{-s}{n+1} L_p\left(s+n+1, t; \chi_{n+1}\right).
\end{aligned}$$

Therefore

$$\frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) = (n+1)! q^{n+1} \binom{-s}{n+1} L_p\left(s+n+1, t; \chi_{n+1}\right),$$

and the lemma must hold by induction. □

With this result, we can derive a more general power series expansion of $L_p(s,t;\chi)$.

THEOREM 4.7. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then for* $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$,

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t-\alpha)^m L_p(s+m,\alpha;\chi_m) .$$

REMARK. Note that Theorem 4.5 is the case of $\alpha = 0$ here.

Proof. It follows from the Taylor series expansion of $L_p(s,t;\chi)$ in the variable t about α (see Proposition 2.6) that we can write $L_p(s,t;\chi)$ in the form

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \beta_m (t-\alpha)^m,$$

where

$$\beta_m = \frac{1}{m!}\frac{\partial^m}{\partial t^m} L_p(s,t;\chi)\bigg|_{t=\alpha} .$$

From Lemma 4.6

$$\frac{1}{m!}\frac{\partial^m}{\partial t^m} L_p(s,t;\chi) = \binom{-s}{m} q^m L_p(s+m,t;\chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s+m,\alpha;\chi_m),$$

completing the proof. $\square$

4.3 RELATING $L_p(s,t;\chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of $L_p(s,t;\chi)$.

For the character χ, let $F_0 = \operatorname{lcm}(f_\chi, q)$. Then $f_{\chi_n} \mid F_0$ for each $n \in \mathbf{Z}$. Also, let F be a positive multiple of $pq^{-1}F_0$.

THEOREM 4.8. *Let $t \in \mathbf{C}_p$, $|t|_p \le 1$, and $s \in \mathfrak{D}$, except $s \ne 1$ if $\chi = 1$. Then*

$$L_p(s, t+F; \chi) - L_p(s, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{-s}. \tag{23}$$

Proof. Let $t \in \mathbf{C}_p$, $|t|_p \le 1$, and let $n \in \mathbf{Z}$, $n \ge 1$. Then from (18),

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\frac{1}{n}\left(b_n(t+F) - b_n(t)\right).$$

Now, (19) implies

$$\begin{aligned} b_n(t+F) - b_n(t) &= \left(B_{n,\chi_n}(q(t+F)) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q(t+F)\right)\right) \\ &\quad - \left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right) \\ &= \left(B_{n,\chi_n}(q(t+F)) - B_{n,\chi_n}(qt)\right) \\ &\quad - \chi_n(p)p^{n-1}\left(B_{n,\chi_n}\left(p^{-1}q(t+F)\right) - B_{n,\chi_n}\left(p^{-1}qt\right)\right). \end{aligned}$$

Thus, by (4), we can write

$$\begin{aligned} &b_n(t+F) - b_n(t) \\ &\quad = n\sum_{a=1}^{qF} \chi_n(a)(a+qt)^{n-1} - n\chi_n(p)p^{n-1}\sum_{a=1}^{p^{-1}qF} \chi_n(a)(a+p^{-1}qt)^{n-1} \\ &\quad = n\sum_{a=1}^{qF} \chi_n(a)(a+qt)^{n-1} - n\sum_{\substack{a=1 \\ p \mid a}}^{qF} \chi_n(a)(a+qt)^{n-1}. \end{aligned}$$

Therefore,

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_n(a)(a+qt)^{n-1}.$$

Now, $\chi_n = \chi_1\omega^{-(n-1)}$, so that

$$\begin{aligned} \chi_n(a)(a+qt)^{n-1} &= \chi_1(a)\omega^{-(n-1)}(a)(a+qt)^{n-1} \\ &= \chi_1(a)\langle a+qt\rangle^{n-1}. \end{aligned}$$

Thus

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{n-1},$$

and (23) holds for all $s = 1 - n$, where $n \in \mathbf{Z}$, $n \geq 1$. Therefore, since the negative integers have 0 as a limit point, Lemma 2.5 implies that Theorem 4.8 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (23).

It is obvious that the domains, in the variable s, of the functions on the left of (23) contain $\mathfrak{D}$, except $s \neq 1$ when $\chi = 1$. Consider now the function

$$-\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{-s} = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{-1}\langle a+qt\rangle^{1-s}.$$

Since it consists of a finite sum of functions of the form $\langle a+qt\rangle^{1-s}$, where $a \in \mathbf{Z}$, $(a,p) = 1$, we need only show that each such function is analytic on $\mathfrak{D}$, and the proof will be complete.

The quantity $\langle a+qt\rangle^{1-s}$ can be written as

$$\langle a+qt\rangle^{1-s} = \exp\left((1-s)\log\langle a+qt\rangle\right),$$

and by (9), the Taylor series expansion of the exponential function,

$$\langle a+qt\rangle^{1-s} = \sum_{m=0}^{\infty} \frac{1}{m!}(1-s)^m \left(\log\langle a+qt\rangle\right)^m.$$

Since $\langle a+qt\rangle \equiv 1 \pmod{q\mathfrak{o}}$ for $a \in \mathbf{Z}$, $(a,p) = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we must also have $\log\langle a+qt\rangle \equiv 0 \pmod{q\mathfrak{o}}$ for such a and t. Thus

$$\left|\frac{1}{m!}(1-s)^m \left(\log\langle a+qt\rangle\right)^m\right|_p \leq \left|\frac{1}{m!}q^m(s-1)^m\right|_p$$

for all m. By (8) we can write

$$\begin{aligned}\left|\frac{1}{m!}q^m(s-1)^m\right|_p &\leq \left|p^{-m/(p-1)}q^m(s-1)^m\right|_p\\ &= \left|p^{-1/(p-1)}q(s-1)\right|_p^m.\end{aligned}$$

Thus if

$$\left|p^{-1/(p-1)}q(s-1)\right|_p < 1,$$

then

$$\left|\frac{1}{m!}(1-s)^m \left(\log\langle a+qt\rangle\right)^m\right|_p \to 0$$

as $m \to \infty$. So whenever $|s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$, meaning that $s \in \mathfrak{D}$, we have convergence for the power series. Therefore, the functions on either side of (23) have domains that contain $\mathfrak{D}$, except possibly for $s = 1$ when $\chi = 1$, and the theorem must hold. $\square$

COROLLARY 4.9. *Let $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-s}.$$

Proof. This follows from Theorem 4.8 since $L_p(s, 0; \chi) = L_p(s; \chi)$ for any character χ. □

We shall now consider how Corollary 4.9 can be utilized to derive a collection of congruences related to the generalized Bernoulli polynomials. Let Δ_c denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$. Repeated application of this operator can be expressed in the form

$$\Delta_c^k x_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

Recall that $F_0 = \operatorname{lcm}(f_\chi, q)$. For $n \in \mathbf{Z}$, $n \geq 1$, denote

$$\beta_{n,\chi}(t) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right).$$

This is the polynomial structure that we utilized with respect to generalizing the p-adic L-functions. We will incorporate this structure in an extension of the Kummer congruences, but the results that we derive will be without restriction on either χ or p.

THEOREM 4.10. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity $q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$, and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.*

Proof. Since Δ_c is a linear operator, Corollary 4.9 implies that

$$\Delta_c^k L_p(1-n, F; \chi) = \Delta_c^k L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\Delta_c^k\langle a\rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Thus

$$\Delta_c^k\beta_{n,\chi}(F) - \Delta_c^k\beta_{n,\chi}(0) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-1}\Delta_c^k\langle a\rangle^{n}.$$

Note that

$$(24)\qquad \Delta_c^k\langle a\rangle^n=\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\langle a\rangle^{n+mc}=\langle a\rangle^n\left(\langle a\rangle^c-1\right)^k .$$

Now, $\langle a\rangle\equiv 1 \pmod{q\mathbf{Z}_p}$, which implies that $\langle a\rangle^c\equiv 1 \pmod{q\mathbf{Z}_p}$, and thus

$$\Delta_c^k\langle a\rangle^n\equiv 0 \pmod{q^k\mathbf{Z}_p}.$$

Therefore

$$\Delta_c^k\beta_{n,\chi}(F)-\Delta_c^k\beta_{n,\chi}(0)\equiv 0 \pmod{q^k\mathbf{Z}_p[\chi]},$$

and so $q^{-k}\Delta_c^k\beta_{n,\chi}(F)-q^{-k}\Delta_c^k\beta_{n,\chi}(0)\in\mathbf{Z}_p[\chi]$. Also, since $\langle a\rangle^n\equiv 1 \pmod{q\mathbf{Z}_p}$,

$$(25)\qquad q^{-k}\Delta_c^k\beta_{n,\chi}(F)-q^{-k}\Delta_c^k\beta_{n,\chi}(0)=-\sum_{\substack{a=1\\(a,p)=1}}^{qF}\chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c-1}{q}\right)^k$$

implies that the value of $q^{-k}\Delta_c^k\beta_{n,\chi}(F)-q^{-k}\Delta_c^k\beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n.

Let $\tau\in pq^{-1}F_0\mathbf{Z}_p$. Since the set of positive integers in $pq^{-1}F_0\mathbf{Z}$ is dense in $pq^{-1}F_0\mathbf{Z}_p$, there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i>0$ for each i, such that $\tau_i\to\tau$. Now, $\beta_{n,\chi}(t)$ is a polynomial, which implies that $\beta_{n,\chi}(\tau_i)\to\beta_{n,\chi}(\tau)$. Therefore

$$\lim_{i\to\infty}\left(\Delta_c^k\beta_{n,\chi}(\tau_i)-\Delta_c^k\beta_{n,\chi}(0)\right)=\Delta_c^k\beta_{n,\chi}(\tau)-\Delta_c^k\beta_{n,\chi}(0).$$

The left side of this equality is 0 modulo $q^k\mathbf{Z}_p[\chi]$, which implies that

$$\Delta_c^k\beta_{n,\chi}(\tau)-\Delta_c^k\beta_{n,\chi}(0)\equiv 0 \pmod{q^k\mathbf{Z}_p[\chi]},$$

and so $q^{-k}\Delta_c^k\beta_{n,\chi}(\tau)-q^{-k}\Delta_c^k\beta_{n,\chi}(0)\in\mathbf{Z}_p[\chi]$. Furthermore, for n' a positive integer,

$$\begin{aligned}\lim_{i\to\infty}&\left(\left(q^{-k}\Delta_c^k\beta_{n,\chi}(\tau_i)-q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right)-\left(q^{-k}\Delta_c^k\beta_{n',\chi}(\tau_i)-q^{-k}\Delta_c^k\beta_{n',\chi}(0)\right)\right)\\&=\left(\left(q^{-k}\Delta_c^k\beta_{n,\chi}(\tau)-q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right)-\left(q^{-k}\Delta_c^k\beta_{n',\chi}(\tau)-q^{-k}\Delta_c^k\beta_{n',\chi}(0)\right)\right).\end{aligned}$$

Since $\tau_i\in pq^{-1}F_0\mathbf{Z}$ for each i, the quantity on the left must also be 0 modulo $q\mathbf{Z}_p[\chi]$. Therefore the value of $q^{-k}\Delta_c^k\beta_{n,\chi}(\tau)-q^{-k}\Delta_c^k\beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n. □

THEOREM 4.11. *Let n, c, k, and k' be positive integers with $k \equiv k' \pmod{p-1}$, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \le |pq^{-1}F_0|_p$. Then*

$$q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}.$$

Proof. Let k and k' be positive integers such that $k \equiv k' \pmod{p-1}$. Without loss of generality, we can assume that $k \ge k'$. From (25),

$$\begin{aligned}
&\left(q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right) - \left(q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)\right) \\
&\quad = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c - 1}{q}\right)^k + \sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c - 1}{q}\right)^{k'} \\
&\quad = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c - 1}{q}\right)^{k'}\left(\left(\frac{\langle a\rangle^c - 1}{q}\right)^{k-k'} - 1\right),
\end{aligned}$$

where F is a positive multiple of $pq^{-1}F_0$. If a is such that

$$\langle a\rangle^c - 1 \not\equiv 0 \pmod{pq\mathbf{Z}_p},$$

then

$$\left(\frac{\langle a\rangle^c - 1}{q}\right)^{k-k'} - 1 \equiv 0 \pmod{p\mathbf{Z}_p},$$

since $k - k' \equiv 0 \pmod{p-1}$. Thus

$$q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}.$$

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Then there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. Consider

$$\begin{aligned}
&\lim_{i\to\infty}\left((q^{-k}\Delta_c^k\beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau_i) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0))\right) \\
&\quad = \left(q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right) - \left(q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)\right).
\end{aligned}$$

Since the left side of this equality must be 0 modulo $p\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

THEOREM 4.12. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity*

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.

Proof. We are once again working with a linear operator, so Corollary 4.9 implies that

$$\binom{q^{-1}\Delta_c}{k}L_p(1-n,F;\chi) = \binom{q^{-1}\Delta_c}{k}L_p(1-n;\chi) - \sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Then

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-1}\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n}.$$

Utilizing (15), we can write

$$\begin{aligned}\binom{q^{-1}\Delta_c}{k}\langle a\rangle^n &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)q^{-m}\Delta_c^m\langle a\rangle^n \\ &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)q^{-m}\langle a\rangle^n\left(\langle a\rangle^c - 1\right)^m,\end{aligned}$$

which follows from (24). This can then be rewritten as

$$\binom{q^{-1}\Delta_c}{k}\langle a\rangle^n = \langle a\rangle^n\binom{q^{-1}(\langle a\rangle^c - 1)}{k}.$$

Since $q^{-1}(\langle a\rangle^c - 1) \in \mathbf{Z}_p$ for each $a \in \mathbf{Z}$ with $(a,p) = 1$, we see that

$$\langle a\rangle^n\binom{q^{-1}(\langle a\rangle^c - 1)}{k} \in \mathbf{Z}_p.$$

This then implies that

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi].$$

Furthermore, since $\langle a\rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$, the value of this quantity modulo $q\mathbf{Z}_p[\chi]$ is independent of n.

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. We are working with polynomials, so that

$$\lim_{i\to\infty}\left(\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) = \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0),$$

which must be in $\mathbf{Z}_p[\chi]$ since the limit of any sequence in $\mathbf{Z}_p[\chi]$ must also be in $\mathbf{Z}_p[\chi]$. Now let n' be a positive integer, and consider

$$\begin{aligned}\lim_{i\to\infty}&\left(\left(\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n,\chi}(\tau_i) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) - \left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(\tau_i) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(0)\right)\right)\\ &= \left(\left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) - \left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(\tau) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(0)\right)\right).\end{aligned}$$

The quantity on the left must be 0 modulo $q\mathbf{Z}_p[\chi]$, which implies that the value of

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)$$

modulo $q\mathbf{Z}_p[\chi]$ is independent of n. $\square$

4.4 Generalized Bernoulli power series

In [9] we find a definition of ordinary Bernoulli numbers of negative index, B_{-n}, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{Q}_p$, given by

$$B_{-n} = \lim_{k\to\infty} B_{\phi(p^k)-n}, \tag{26}$$

where the limit is taken in a p-adic sense. Note that $\phi(p^k) \to 0$ in $\mathbf{Z}_p$ as $k \to \infty$. Since $|B_m|_p$ is bounded for all $m \in \mathbf{Z}$, $m \geq 0$, we must have

$$\begin{aligned}B_{-n} &= \lim_{k\to\infty}\left(1 - p^{\phi(p^k)-n-1}\right)B_{\phi(p^k)-n}\\ &= \lim_{k\to\infty} -\left(\phi\left(p^k\right) - n\right)L_p\left(1 - \left(\phi\left(p^k\right) - n\right); \omega^{-n}\right)\\ &= nL_p\left(n+1; \omega^{-n}\right).\end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that $B_m = 0$ for any odd $m \in \mathbf{Z}$, $m \geq 3$. Thus (26) implies that $B_{-n} = 0$ for any odd $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we have the following:

THEOREM 4.13. *Let $n \in \mathbf{Z}$ be even, $n \geq 2$. Then*

$$B_{-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p ,$$

where each prime r is taken to be a rational prime.

REMARK. Since $1/r \in \mathbf{Z}_p$ for any rational prime $r \neq p$, this implies that $B_{-n} + 1/p \in \mathbf{Z}_p$ whenever $(p-1) \mid n$, and $B_{-n} \in \mathbf{Z}_p$ otherwise.

Proof. By the von Staudt-Clausen theorem, we know that

$$B_m + \sum_{\substack{r \text{ prime} \\ (r-1)|m}} \frac{1}{r} \in \mathbf{Z}$$

for any even $m \in \mathbf{Z}$, $m \geq 2$.

Let $n \in \mathbf{Z}$ be even, $n \geq 2$. For any integer $k \geq 2$, $\phi(p^k)$ is even and $(p-1) \mid \phi(p^k)$. Thus $\phi(p^k) - n$ is even, and $(p-1) \mid n$ if and only if $(p-1) \mid (\phi(p^k) - n)$. Therefore, if k is sufficiently large,

$$B_{\phi(p^k)-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p ,$$

and the result follows from (26). $\square$

In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n,\chi}$, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{C}_p$ according to

$$B_{-n,\chi} = \lim_{k \to \infty} B_{\phi(p^k)-n,\chi} , \tag{27}$$

where the limit is once again taken in a p-adic sense. For each $m \in \mathbf{Z}$, $m \geq 0$, the quantity $|B_{m,\chi}|_p$ is bounded. Thus, since $\chi_{\phi(p^k)} = \chi$ for all characters χ and for all $k \in \mathbf{Z}$, $k \geq 1$, we can write

$$\begin{aligned} B_{-n,\chi} &= \lim_{k \to \infty} \left(1 - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n-1}\right) B_{\phi(p^k)-n,\chi_{\phi(p^k)}} \\ &= \lim_{k \to \infty} -\left(\phi\left(p^k\right) - n\right) L_p\left(1 - \left(\phi\left(p^k\right) - n\right); \chi_n\right) \\ &= nL_p\left(n+1; \chi_n\right), \end{aligned}$$

so that the limit exists. Since $B_{\phi(p^k)-n,1} = B_{\phi(p^k)-n}$ for $n, k \in \mathbf{Z}$, with $n \geq 1$ and k sufficiently large, we obtain $B_{-n,1} = B_{-n}$ for all such n.

If $k \geq 2$, then $\phi(p^k)$ is even. Thus n and $\phi(p^k) - n$ are of the same parity. Recall that

$$\delta_\chi = \begin{cases} 1, & \text{if } \chi \text{ is odd} \\ 0, & \text{if } \chi \text{ is even}. \end{cases}$$

Then $B_{\phi(p^k)-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod 2$, provided $\phi(p^k) - n > 1$. Because of this, the relation (27) implies that $B_{-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod 2$ for all $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we can obtain

THEOREM 4.14. *Let χ be such that $\chi \neq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $f_\chi B_{-n,\chi} \in \mathbf{Z}_p[\chi]$.*

Proof. Recall that when $\chi \neq 1$, $f_\chi B_{m,\chi} \in \mathbf{Z}[\chi]$ for all $m \in \mathbf{Z}$, $m \geq 0$. Thus

$$f_\chi B_{-n,\chi} = \lim_{k\to\infty} f_\chi B_{\phi(p^k)-n,\chi}$$

must be in the p-adic completion of $\mathbf{Z}[\chi]$ for any $n \in \mathbf{Z}$, $n \geq 1$. Since the p-adic completion of $\mathbf{Z}[\chi]$ is $\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

We now define what we shall refer to as generalized Bernoulli power series of negative index in $\mathbf{Z}_p[\chi]$. For $n \in \mathbf{Z}$, $n \geq 1$, and for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$, let

$$B_{-n,\chi}(t) = \lim_{k\to\infty} B_{\phi(p^k)-n,\chi}(t).$$

Then

$$\begin{aligned} B_{-n,\chi}(qt) &= \lim_{k\to\infty} \left(B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(qt) - \chi_{\phi(p^k)}(p)p^{\phi(p^k)-n-1}B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(p^{-1}qt)\right) \\ &= \lim_{k\to\infty} -(\phi(p^k) - n)L_p(1 - (\phi(p^k) - n), t; \chi_n) \\ &= nL_p(n+1, t; \chi_n). \end{aligned}$$

Since $L_p(n+1, t; \chi_n)$ exists for each $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we see that $B_{-n,\chi}(qt)$ must also exist for such t. Thus $B_{-n,\chi}(t)$ exists for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$. Now, by Theorem 4.5, we can expand this quantity as a power series, obtaining

$$\begin{aligned} B_{-n,\chi}(qt) &= n\sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m L_p\left(n+m+1; \chi_{n+m}\right) \\ &= n\sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m \frac{1}{n+m} B_{-(n+m),\chi} \\ &= \sum_{m=0}^{\infty} \binom{-n}{m} B_{-(n+m),\chi} q^m t^m. \end{aligned}$$

Since $|B_{-(n+m),\chi}|_p \leq \max\{|p|_p^{-1}, |f_\chi|_p^{-1}\}$ and

$$\binom{-n}{m} = (-1)^m \binom{n+m-1}{m},$$

this sum converges for $|qt|_p < 1$. Thus we have the relation

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m, \tag{28}$$

converging for all $t \in \mathbf{C}_p$, $|t|_p < 1$. Note that this is in the same form as (2) for the generalized Bernoulli polynomials having positive index, which we can rewrite as

$$B_{n,\chi}(t) = \sum_{m=0}^{\infty} \binom{n}{m} B_{n-m,\chi} t^m,$$

since $\binom{n}{m} = 0$ for $m, n \in \mathbf{Z}$, $m > n \geq 0$. By setting $t = 0$ in (28), we see that $B_{-n,\chi}(0) = B_{-n,\chi}$ for all $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.15. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for any $m \in \mathbf{Z}$, $m \geq 1$, such that $q \mid mf_\chi$,*

$$B_{-n,\chi}\left(mf_\chi\right) - B_{-n,\chi}(0) = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1}.$$

Proof. By definition, since $|mf_\chi|_p \leq |q|_p$,

$$\begin{aligned} B_{-n,\chi}\left(mf_\chi\right) - B_{-n,\chi}(0) &= \lim_{k\to\infty} \left(B_{\phi(p^k)-n,\chi}\left(mf_\chi\right) - B_{\phi(p^k)-n,\chi}(0)\right) \\ &= \lim_{k\to\infty} \left(\phi\left(p^k\right) - n\right) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1}, \end{aligned}$$

following from (4). Now, $v_p(\phi(p^k)) = k - 1$, and $a^{\phi(p^k)} \equiv 1 \pmod{p^k}$ for $(a,p) = 1$. These imply that

$$\lim_{k\to\infty} \left(\phi\left(p^k\right) - n\right) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1} = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1},$$

completing the proof. $\square$

THEOREM 4.16. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for all χ and for all $t \in \mathbf{C}_p$, $|t|_p < 1$,*

$$B_{-n,\chi}(-t) = (-1)^n \chi(-1) B_{-n,\chi}(t)\,.$$

Proof. Since

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m,$$

and $B_{-n-m,\chi} = 0$ whenever $n+m \not\equiv \delta_\chi \pmod 2$ for each $m \in \mathbf{Z}$, $m \geq 1$, we see that $B_{-n,\chi}(t)$ is either an odd or an even function according to whether $n + \delta_\chi$ is odd or even, respectively. Thus

$$\begin{aligned} B_{-n,\chi}(-t) &= (-1)^{n+\delta_\chi} B_{-n,\chi}(t) \\ &= (-1)^n \chi(-1) B_{-n,\chi}(t)\,, \end{aligned}$$

and the proof is complete. □

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