5. Teichmüller space

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5. TEICHMÜLLER SPACE

DEFINITION. The *space* $\mathcal{P}(g)$ *of canonical polygons* contains all canonical polygons $P(g)$ with the topology $P_j(g) \to P(g)$ if and only if the lengths of all sides converge and all angles converge, more precisely, if and only if

$$a_i(P_j(g)) \to a_i(P(g)), \quad i = 1, \ldots, 4g\,,$$

(where $a_i(P_j(g))$ is the side a_i of $P_j(g)$) and

$$\alpha_i(P_j(g)) \to \alpha_i(P(g)), \quad i = 1, \ldots, 4g\,,$$

(where $\alpha_i(P_j(g))$ is the angle α_i of $P_j(g)$).

REMARKS. (i) Note that two canonical polygons $P(g)$ and $P'(g)$ may be isometric, but represent different points in $\mathcal{P}(g)$. They represent the same point if and only if there is an isometry mapping the side $a_i(P(g))$ to the side $a_i(P'(g))$, $i = 1, \ldots, 4g$ (and not to the side $a_j(P'(g))$, $j \neq i$). One expresses this fact by saying that the sides of the canonical polygons are *marked*.

(ii) One may calculate the dimension of $\mathcal{P}(g)$ in the following heuristic way (this argument is modeled after one given in [16]). A canonical polygon has $4g$ vertices. Each vertex is determined in $\mathbf{H}$ by two (real) parameters, this gives $8g$ parameters. The dimension of the space of isometries of $\mathbf{H}$ is 3 so we remain with $8g - 3$ parameters. By condition (I) of a canonical polygon we have $2g$ equalities and each of the conditions (II), (IV), (V) gives one equality. We remain with

$$8g - 3 - 2g - 3 = 6g - 6$$

parameters.

THEOREM 11. *$\mathcal{P}(g)$ is homeomorphic to* $\mathbf{R}^{6g-6}$.

REMARK. The following proof is new. The theorem was first proved by Coldewey and Zieschang in an annex to [17], see also [18]. An (indirect) proof has also been given by Buser [2], compare the introduction.

Proof. (i) Let $P(g)$ be a canonical polygon with sides a_i and angles α_i between a_i and a_{i+1}, $i = 1, \ldots, 4g$ (the indices are taken modulo $4g$). Let $\{Q_i\} = a_i \cap a_{i+1}$, $i = 1, \ldots, 4g$. Denote by b_i the geodesic segment between

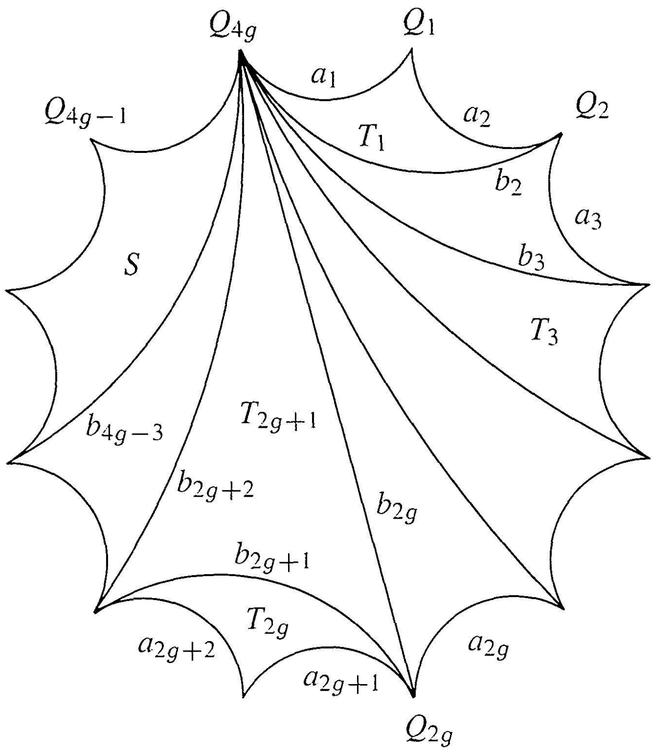

FIGURE 5
The "triangulation" of a canonical polygon $P(g)$

Q_{4g} and Q_i, $i = 2, \ldots, 4g-3$, $i \neq 2g+1$. Denote by b_{2g+1} the geodesic segment between Q_{2g} and Q_{2g+2}, compare Figure 5.

$P(g)$ is separated by the geodesic segments $b_2, \ldots, b_{4g-3}$ into one quadrilateral S and $4g-4$ triangles T_i, $i = 1, \ldots, 4g-4$, with sides b_i, b_{i+1}, a_{i+1} for $i = 2, \ldots, 4g-4$, $i \neq 2g$, $i \neq 2g+1$; the triangle T_1 has sides a_1, a_2, b_2, the triangle T_{2g} has sides $a_{2g+1}, a_{2g+2}, b_{2g+1}$, and the triangle T_{2g+1} has sides $b_{2g}, b_{2g+1}, b_{2g+2}$ (note that T_{2g+1} is only defined if $g > 2$).

A point $x = (x_1, \ldots, x_{6g-5}) \in \mathbf{R}^{6g-5}$ is called *admissible* if $x_j > 0$, $j = 1, \ldots, 6g-5$, and if, putting

$$L(a_i) = L(a_{i+2g}) = x_i, \quad i = 1, \ldots, 2g \quad (L = \text{length})$$

and

$$L(b_2) = L(b_{2g+1}) = x_{2g+1}$$

and

$$L(b_i) = x_{2g+i-1}, \quad i = 3, \ldots, 2g; \quad L(b_i) = x_{2g+i-2}, \quad i = 2g+2, \ldots, 4g-3,$$

the triangle inequalities hold for the triangles T_k, $k = 1, \ldots, 4g-4$, and the "quadrilateral inequalities" hold for S (which means that the sum of the lengths of any three sides of S is greater than the length of the fourth side). Note that these are purely algebraic conditions on $x \in \mathbf{R}^{6g-5}$.

Let O be the subset of $\mathbf{R}^{6g-5}$ of admissible points. Being the intersection of a finite number of open sets, O is open. Moreover, O is convex since O is the intersection of a finite number of convex sets, namely, if for example $x_1 + x_2 > x_3$ and $y_1 + y_2 > y_3$, then

$$\lambda(x_1 + x_2) + (1 - \lambda)(y_1 + y_2) > \lambda x_3 + (1 - \lambda)(y_3), \ \forall \lambda \in [0, 1].$$

(ii) Let $x \in O$. Then we associate a formal polygon $P(x)$ to x in the following way. $P(x)$ is the formal union of the triangles $T_k(x)$, $k = 1, \ldots, 4g-4$, and the quadrilateral $S(x)$ in the same way as $P(g)$. Hereby, the triangles, as well as the lengths of the sides of $S(x)$ are defined by the identifications described in part (i). The angles of the triangles are determined by their sides (by Theorem 6). The (formal) angles α_i of $P(x)$, $i = 1, \ldots, 4g$, are defined as the sum of the angles of the corresponding triangles and (if $i \in \{4g-3, 4g-2, 4g-1, 4g\}$) of $S(x)$. Thereby, the angles of $S(x)$ are defined by the conditions that $S(x)$ is convex and that

$$\left| \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^{g} \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal, this minimum is denoted by $\mathbf{m}(x)$. By Corollary 10 the angles of $S(x)$ are then determined and hence also the angles of $P(x)$. Note however that an angle α_i of $P(x)$ may be greater than 2π, this is why $P(x)$ is called a formal polygon with formally defined angles.

(iii) Let $x \in O$. Then tx (for $t \in \mathbf{R}$, $t > 0$) is also in O (since the triangle inequalities remain true). I claim that there exists a unique $t_0 > 0$ (depending on x) such that $P(t_0 x)$ is a canonical polygon. I first show uniqueness. Assume that $\mathbf{m}(tx) > 0$ for $P(tx)$. This means that $\mathbf{A}(tx) - \mathbf{B}(tx) \neq 0$ where

$$\mathbf{A}(tx) := \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} \quad \text{and} \quad \mathbf{B}(tx) := \sum_{i=1}^{g} \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}.$$

If $\mathbf{A}(tx) - \mathbf{B}(tx) > 0$, then an angle in $S(tx)$ must be π and, by Corollary 8 and the minimality of $\mathbf{m}(x)$, this angle must appear in the sum $\mathbf{B}(tx)$. This implies that

$$\Sigma(tx) := \mathbf{A}(tx) + \mathbf{B}(tx) > 2\pi. \tag{5}$$

Of course, (5) also holds if $\mathbf{A}(tx) - \mathbf{B}(tx) < 0$. It follows that if $P(t_0 x)$ is a canonical polygon, then $\mathbf{m}(t_0 x) = 0$ (since $\Sigma(t_0 x) = 2\pi$ by the definition of canonical polygons). Now assume that $P(t_0 x)$ and $P(t_1 x)$ are canonical polygons with $t_1 > t_0$. By Lemma 9, all angles of the triangles $T_k(t_1 x)$

are smaller than the corresponding angles in $T_k(t_0x)$, $k = 1, \dots, 4g-4$. Moreover, by Corollary 10, at least two opposite angles in $S(t_1x)$ are smaller than the corresponding angles in $S(t_0x)$. This implies that $\mathbf{A}(t_1x) < \mathbf{A}(t_0x)$ or $\mathbf{B}(t_1x) < \mathbf{B}(t_0x)$. But since $\mathbf{A}(t_1x) = \mathbf{B}(t_1x)$ and $\mathbf{A}(t_0x) = \mathbf{B}(t_0x)$ ($\mathbf{m}(t_0x) = \mathbf{m}(t_1x) = 0$), it follows that $\Sigma(t_1x) < \Sigma(t_0x)$, a contradiction. This proves uniqueness.

As for existence note that if $t \to 0$, then the volume of all triangles T_k, $k = 1, \dots, 4g-4$, and the volume of S tend to zero which implies by Theorem 3 that

$$\Sigma := \sum_{i=1}^{4g} \alpha_i \to (4g-2)\pi .$$

On the other hand, for $t \to \infty$, all angles in the triangles T_k, $k = 1, \dots, 4g-4$, converge to zero by Lemma 9 and, by Corollary 10(ii), at least two opposite angles of S converge to zero. It follows by the condition that

$$\left| \sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} - \sum_{i=1}^{g} \alpha_{2i} - \sum_{i=g+1}^{2g} \alpha_{2i-1} \right|$$

is minimal that all angles of S converge to zero and hence Σ converge to zero. Therefore, there exists a t_0 such that $\Sigma(t_0x) = 2\pi$. Now $P(t_0x)$ is a canonical polygon. Namely, conditions (I), (II) and (IV) hold by construction. By the argument above, we further have $\mathbf{m}(t_0x) = 0$ and condition (V) holds. Finally, condition (III) holds since all sides of the triangles of $P(t_0x)$ have finite length and since conditions (II) and (V) hold.

(iv) We therefore have defined a projection from the open convex set O to the unit sphere in $\mathbf{R}^{6g-5}$. Since all operations are controlled by the formulas of Theorem 6, it is clear that this map is continuous and that the image is homeomorphic to $\mathbf{R}^{6g-6}$ as well as homeomorphic to $\mathcal{P}(g)$ since every canonical polygon is thereby obtained. $\square$

DEFINITION. By Theorem 5 each of the canonical polygons in $\mathcal{P}(g)$ defines a closed hyperbolic surface of genus g. The *Teichmüller space* T_g is the space of these hyperbolic surfaces with the topology induced from that of $\mathcal{P}(g)$.

COROLLARY 12. *T_g is homeomorphic to* $\mathbf{R}^{6g-6}$. $\square$