2. Hyperbolic geometry and Fuchsian groups

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canonical polygons. Section 4 provides the necessary material from hyperbolic trigonometry, it contains also some lemmas needed later. Section 5 contains the proof of the main theorem and Section 6 gives some applications, mainly concerning hyperelliptic Riemann surfaces. More precisely, I give a new proof of a geometric characterization of hyperelliptic Riemann surfaces which first appeared in [14] (I thank very much Feng Luo who, by his comments on [14], has contributed to the idea of this new proof). I also show (and this is a new result) that the Teichmüller space T_g for $g=2$ can be parametrized by 7 geodesic length functions, taken as homogeneous parameters. This is the optimum parametrization of Teichmüller space by geodesic length functions which one can expect.

I spoke about the content of this paper in lectures of the Troisième Cycle Romand de Mathématiques (Lausanne 1997); I thank the participants for their comments.

2. Hyperbolic geometry and Fuchsian groups

The material of this section and of parts of the following section is standard, see for example [1], [4], [5], [6], [7], [8], [15].

DEFINITION. (i) $\mathbf{H} = \{z = (x,y) \in \mathbf{C} : y > 0\}$ denotes the *upper halfplane.* The *hyperbolic metric* on $\mathbf{H}$ is given by

$$dz = \frac{1}{y}(dz)_E$$

where $(dz)_E$ is the standard Euclidean metric on $\mathbf{C}$ and y is the imaginary part of z.

(ii) Define

$$\mathrm{SL}(2,\mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1;\ a,b,c,d \in \mathbf{R} \right\}$$

and

$$\mathrm{PSL}(2,\mathbf{R}) = \mathrm{SL}(2,\mathbf{R})/\!\sim$$

with $A \sim B$ if and only if $A = \pm B$ for $A, B \in \mathrm{SL}(2,\mathbf{R})$. Let $\gamma \in \mathrm{SL}(2,\mathbf{R})$,

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the action of γ on $\mathbf{H}$ is defined as

$$\gamma(z) = \frac{az+b}{cz+d}$$

for $z \in \mathbf{H}$.

THEOREM 1. $\mathbf{H}$ *is a complete Riemannian manifold of constant curvature* -1. *The geodesics in* $\mathbf{H}$ *are either Euclidean semicircles which are orthogonal to the real axis or vertical half-lines.*

THEOREM 2.

(i) $\mathrm{PSL}(2,\mathbf{R}) = Isom^{+}(\mathbf{H})$, *the group of orientation preserving isometries of* $\mathbf{H}$.

(ii) *Let* u *and* v *be geodesics in* $\mathbf{H}$, *let* z *be on* u *and* z' *on* v. *Then there exists* $\gamma \in \mathrm{PSL}(2,\mathbf{R})$ *with* $\gamma(u) = v$ *and* $\gamma(z) = z'$.

DEFINITION. For a measurable subset $G \subset \mathbf{H}$ define the *volume* $vol(G)$ as

$$vol(G) = \int_G \frac{dx\,dy}{y^2}.$$

REMARK. The volume is invariant under $\gamma \in \mathrm{SL}(2,\mathbf{R})$.

CONVENTIONS. (i) Speaking of triangles, quadrilaterals and polygons always means that the sides are hyperbolic geodesic segments in $\mathbf{H}$.

(ii) Speaking of *angles* in triangles, quadrilaterals and polygons always means *interior angles*.

THEOREM 3. *The volume of a polygon with angles* α_i, $i = 1, 2, \ldots, m$, $m \geq 3$, *is*

$$(m-2)\pi - \sum_{i=1}^{m} \alpha_i.$$

DEFINITION. A *Fuchsian group* Γ is a *discrete* subgroup of $\mathrm{PSL}(2,\mathbf{R})$ where discrete means that the identity matrix is not a cluster point in Γ with respect to the topology induced by the standard topology of $\mathbf{R}^4$.

THEOREM 4. *Let Γ be a Fuchsian group without elliptic elements (an element $\gamma \in \mathrm{PSL}(2, \mathbf{R})$ is elliptic if $|\mathrm{tr}(\gamma)| < 2$ where* tr *is the trace). Then $\mathbf{H}/\Gamma$ is a complete connected orientable Riemannian manifold of dimension 2 with a metric of constant curvature -1.*

DEFINITION. A *hyperbolic surface* is a connected orientable manifold $M = \mathbf{H}/\Gamma$ as in Theorem 4 (where Γ is a Fuchsian group without elliptic elements). M is called *closed* if M is compact and has no boundary.

3. FUNDAMENTAL DOMAINS AND CANONICAL POLYGONS

DEFINITION (Compare Figure 2). Let $g \geq 2$ be an integer. A *canonical polygon* $P(g)$ is a polygon with $4g$ sides, denoted by $a_1, \ldots, a_{4g}$, ordered clockwise, and angles α_i between a_i and a_{i+1}, $i = 1, \ldots, 4g$ (indices are taken modulo $4g$), such that

(I) a_i and a_{i+2g} have the same length, $i = 1, \ldots, 2g$;

(II) the sum of the angles of $P(g)$ is 2π;

(III) $0 < \alpha_i < \pi$, $i = 1, \ldots, 4g$;

(IV) $\alpha_1 = \alpha_{2g+1}$;

(V) $\displaystyle\sum_{i=1}^{g} \alpha_{2i-1} + \sum_{i=g+1}^{2g} \alpha_{2i} = \sum_{i=1}^{g} \alpha_{2i} + \sum_{i=g+1}^{2g} \alpha_{2i-1}$.

I shall speak of condition (I) (or (II) or (III) or (IV) or (V)) referring to this definition.

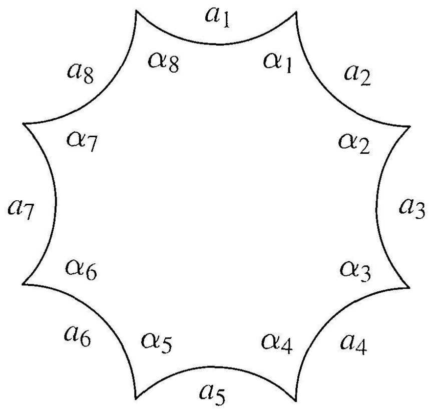

FIGURE 2
A canonical polygon $P(g)$ for $g = 2$