6. Second proof of Theorem 2.4

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PROPOSITION 5.6 ([Ami80, Equation 4.4)]. *Let X be a power series in t over a matrix ring, such that $X(0)=\mathbf{1}$. Then*

$$\det X = \exp\Big(-\int \operatorname{tr}\Big(\frac{\mathbf{1}-X}{Xt}\Big)dt\Big)\,,$$

where the integration is the formal linear operation on power series that maps t^r to $t^{r+1}/(r+1)$.

We then have, using Lemmas 5.2 and 5.3,

$$\begin{aligned}
\frac{\det M}{(1+(1-u)t)^n(1-(1-u)^2t^2)^m} &= \det\frac{M}{\mathbf{1}+(1-u)Jt}\\
&= \exp\Big(-\int \operatorname{tr}\frac{\mathbf{1}+(1-u)Jt-M}{Mt}dt\Big)\\
&= \exp\Big(-\int \begin{array}{l}\text{series counting non-trivial circuits,}\\ \text{length shifted down by one}\end{array} dt\Big)\\
&= \exp\Big(-\int \operatorname{tr}\frac{(1-(1-u)^2t^2)\mathbf{1}-P}{Pt}dt\Big)\\
&= \det\frac{P}{1-(1-u)^2t^2} = \frac{\det P}{(1-(1-u)^2t^2)^{|V(\mathcal{X})|}}\,.
\end{aligned}$$

6. SECOND PROOF OF THEOREM 2.4

Let $P=[\star,\dagger]$ be the set of paths in $\mathcal{X}$ from $\star$ to $\dagger$. As we shall apply the principle of inclusion-exclusion [Wil90], it will be helpful to compute in $\Pi=\mathbf{Z}[[P]]$, the $\mathbf{Z}$-module of functions from the set of paths to $\mathbf{Z}$. We embed subsets of P in Π by mapping a subset to its characteristic function:

$$P\supset A\mapsto \chi_A\,,\quad \text{with } (\pi)\chi_A=\begin{cases}1 & \text{if } \pi\in A\,,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\mathcal{B}$ be the subset of bounded non-negative elements of Π (i.e. functions f such that there is a constant N with $0\le(\pi)f<N$ for all paths π). If ℓ is a complete labelling of $\mathcal{X}$, there is an induced labelling $\ell_*\colon\mathcal{B}\to\mathbf{k}$ given by

$$(f)\ell_* = \sum_{\pi\in P}(\pi)f\pi^\ell\,.$$

Note that the sum, although infinite, defines an element of $\mathbf{k}$ due to the fact that ℓ is complete.

DEFINITION 6.1 (Bump Scheme). Let $e \in E(\mathcal{X})$ and $v \in V(\mathcal{X})$. A *squiggle along* e is a sequence $(e, \bar{e}, \dots, e, \bar{e})$. A *squiggle at* v is a squiggle along e for some edge e such that $e^\alpha = v$.

Let $\pi = (v_0, e_1, \dots, e_n, v_n)$ be a path of length n in $\mathcal{X}$. A *bump scheme* for π is a pair $B = \big((\beta_0, \dots, \beta_n), (\gamma_1, \dots, \gamma_n)\big)$, with

- for all $i \in \{0, \dots, n\}$, a finite (possibly empty) sequence $\beta_i = (\beta_{i,1}, \dots, \beta_{i,t_i})$ of squiggles at v_i;
- for all $i \in \{1, \dots, n\}$, a squiggle γ_i along e_i.

The *weight* $|B|$ of the bump scheme B is defined as

$$|B| = \sum_{i=0}^{n} \sum_{j=1}^{t_i} (|\beta_{i,j}| - 1) + \sum_{i=1}^{n} |\gamma_i| .$$

Given a path π and a bump scheme $B = (\beta, \gamma)$ for π, we obtain a new path $\pi \vee B \in P$, by setting

$$\pi \vee B = \beta_{0,1} \cdot \dots \cdot \beta_{0,t_0} \gamma_1 e_1 \beta_{1,1} \cdot \dots \cdot \gamma_n e_n \beta_{n,1} \cdot \dots \cdot \beta_{n,t_n} ,$$

where the product denotes concatenation.

We now define a linear map $\phi \colon \Pi \to \Pi[[u]]$ by setting, for $f \in \Pi$ and $\pi \in P$,

$$(\pi)\big((f)\phi\big) = \sum_{(\rho,B)\,:\,\rho \vee B = \pi} (u-1)^{|B|}(\rho)f ,$$

where the sum ranges over all pairs (ρ, B) where $\rho \in P$ and B is a bump scheme for ρ such that $\rho \vee B = \pi$. Note that the sum is finite because the edges of ρ and of B form subsets of those of π.

LEMMA 6.2. *For any path* π *we have*

$$(\pi)\big((\chi_P)\phi\big) = u^{\mathrm{bc}(\pi)} . \tag{6.1}$$

Proof. Say $\pi = (\pi_1, \dots, \pi_n)$ has $m \geq 0$ bumps, at indices $b_1, \dots, b_m$ so that $\pi_{b_i} = \overline{\pi_{b_{i+1}}}$. We will show that the evaluation at π of the left-hand side of (6.1) yields u^m.

We claim there is a bijection between the subsets C of $\{1, \dots, m\}$ and the pairs (ρ_C, B_C) where ρ_C is a path and B_C is a bump scheme for ρ_C with $\pi = \rho_C \vee B_C$; and further $|B_C| = |C|$.

First, take a ρ and a $B = (\beta, \gamma)$ such that $\rho \vee B = \pi$. The path $\rho \vee B$ is obtained by shuffling together the edges of ρ and B, and this partitions the edges of π in two classes, namely *(i)* those coming from ρ and *(ii)* those coming from β and γ. Let $C \subset \{1,\dots,m\}$ be the indices of the bumps b_i in π coming from B, i.e. such that π_{b_i} and $\pi_{b_{i+1}}$ belong to the class *(ii)*. One direction of the bijection is then $(\rho, B) \mapsto C$.

Conversely, given a subset C consider the set $D = \{b_i \mid i \in C\}$. Split it in maximal-length runs of consecutive integers $D = D_1 \sqcup \cdots \sqcup D_t$. For all runs D_i do the following: to $D_i = \{j, j+1, \dots, j+2k-1\}$ of even cardinality associate a squiggle γ_j of length $2k$ along π_j; to $D_i = \{j, j+1, \dots, j+2k-2\}$ of odd cardinality associate a squiggle $\beta_{j,l}$ of length $2k$ at v_{j-1}; then delete in π the edges $\pi_j, \dots, \pi_{j+2k-1}$. This process constructs a bump scheme $B = (\beta, \gamma)$ while pruning edges of π, giving a path γ with $\gamma \vee B = \pi$. These two constructions are inverses, proving the claimed bijection.

It now follows that

$$(\pi)(\chi_P)\phi = \sum_{C \in \{1,\dots,m\}} (u-1)^{|B_C|} = \sum_{r=0}^{m} (u-1)^r \binom{m}{r} = u^m . \qquad \square$$

Let $\ell' \colon E(\mathcal{X}) \to \mathbf{k}[[u]]$ be defined by

$$e^{\ell'} = \frac{1}{1-(e\,\overline{e})^{\ell}(1-u)^2} e^{\ell} K_{e^{\omega}} .$$

We prove Theorem 2.4 by noting that $\mathfrak{G}(\ell) = (\chi_P)\ell_*$, that $\mathfrak{F}(\ell) = (\chi_P\phi)\ell_*$, and that for any $f \in \Pi$ we have $(f\phi)\ell_* = K_\star(f)\ell'_*$. To prove this last equality, take a path $\pi = (\pi_1, \dots, \pi_n)$ on vertices $v_0, \dots, v_n$. Then

$$(\chi_{\{\pi\}}\phi)\ell_* = \sum_B (u-1)^{|B|}(\pi \vee B)^{\ell} ,$$

where the sum ranges over all bump schemes for π, and

$$K_\star \pi^{\ell'} = K_{v_0} \frac{1}{1-(u-1)^2(\pi_1\overline{\pi_1})^{\ell}} \pi_1^{\ell} K_{v_1} \cdot \dots \cdot \frac{1}{1-(u-1)^2(\pi_n\overline{\pi_n})^{\ell}} \pi_n^{\ell} K_{v_n} .$$

It is clear these last two lines are equal; for the power series expansion of the K_{v_i} correspond to all the possible squiggle sequences β_i at v_i, and the power series expansion of the $1/(1-(u-1)^2(\pi_i\overline{\pi_i})^{\ell})$ correspond to all possible squiggles γ_i along π_i.