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Finally in Section 9 we show how to compute the circuit series of a free product of graphs (an analogue of the free products of groups, *via* their Cayley graph), and in Section 10 do the same for direct products of graphs.

3. Applications to other fields

The original motivation for Formula 2.2 was its implication of a well-known result in the theory of random walks on discrete groups.

3.1 Applications to random walks on groups

In this section we show how G is related to random walks and F to cogrowth. This will give a generalization of the main formula (1.1) to homogeneous spaces Π/Ξ, where Ξ does not have to be normal and Π is a free product of infinite-cyclic and order-two groups. For a survey on the topic of random walks see [MW89,Woe94].

Throughout this subsection we will have $F(t) = F(0,t)$. We recall the notion of growth of groups:

Definition 3.1. Let Γ be a group generated by a finite symmetric set S. For a $\gamma \in \Gamma$ define its *length*

$$|\gamma| = \min\{n \in \mathbf{N} : \gamma \in S^n\} .$$

The *growth series* of (Γ, S) is the formal power series

$$f_{(\Gamma,S)}(t) = \sum_{\gamma\in\Gamma} t^{|\gamma|} \in \mathbf{N}[[t]] .$$

Expanding $f_{(\Gamma,S)}(t) = \sum f_n t^n$, the *growth* of (Γ, S) is

$$\alpha(\Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n}$$

(this supremum-limit is actually a limit and is smaller than $|S| - 1$).

Let R be a subset of Γ. The *growth series* of R relative to (Γ, S) is the formal power series

$$f^R_{(\Gamma,S)}(t) = \sum_{\gamma\in R} t^{|\gamma|} \in \mathbf{N}[[t]] .$$

Expanding $f^R_{(\Gamma,S)}(t) = \sum f_n t^n$, define the *growth* of R relative to (Γ, S) as

$$\alpha(R; \Gamma, S) = \limsup_{n\to\infty} \sqrt[n]{f_n} .$$

If X is a transitive right Γ-set, the *simple random walk* on (X,S) is the random walk of a point on X, having probability $1/|S|$ of moving from its current position x to a neighbour $x \cdot s$, for all $s \in S$. Fix a point $\star \in X$, and let p_n be the probability that a walk starting at $\star$ finish at $\star$ after n moves. We define the *spectral radius* (which does not depend on the choice of $\star$) of the random walk as

$$\nu(X,S) = \limsup_{n \to \infty} \sqrt[n]{p_n} \ .$$

A group Π is *quasi-free* if it is a free product of cyclic groups of order 2 and ∞. Equivalently, there exists a finite set S and an involution $\bar{\cdot} : S \to S$ such that, as a monoid,

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle \ .$$

Π is then said to be *quasi-free on* S. All quasi-free groups on S have the same Cayley graph, which is a regular tree of degree $|S|$.

Every group Γ generated by a symmetric set S is a quotient of a quasi-free group in the following way: let $\bar{\cdot}$ be an involution on S such that for all $s \in S$ we have the equality $\bar{s} = s^{-1}$ in Γ. Then Γ is a quotient of the quasi-free group $\langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle$.

The *cogrowth series* (respectively *cogrowth*) of (Γ, S) is defined as the growth series (respectively growth) of $\ker(\pi \colon \Pi \to \Gamma)$ relative to (Π, S), where Π is a quasi-free group on S.

Associated with a group Π generated by a set S and a subgroup Ξ of Π, there is a $|S|$-regular graph $\mathcal{X}$ on which Π acts, called the *Schreier graph* of (Π, S) relative to Ξ. It is given by $\mathcal{X} = (V, E)$, with

$$V = \Xi \backslash \Pi$$

and

$$E = V \times S, \quad (v,s)^\alpha = v, \quad (v,s)^\omega = vs, \quad \overline{(v,s)} = (vs, s^{-1});$$

i.e. two cosets A, B are joined by at least one edge if and only if $AS \supset B$. (This is the Cayley graph of (Π, S) if $\Xi = 1$.) There is a circuit in $\mathcal{X}$ at every vertex $\Xi v \in \Xi\backslash\Pi$ such that $s \in v^{-1}\Xi v$ for some $s \in S$; and there is a multiple edge from Ξv to Ξw in $\mathcal{X}$ if there are $s, t \in v^{-1}\Xi w$ with $s \neq t \in S$.

COROLLARY 3.2 (of Corollary 2.6). *Let Π be a quasi-free group, presented as a monoid as*

$$\Pi = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle .$$

Let $\Xi < \Pi$ be a subgroup of Π. Let $\nu = \nu(\Xi \backslash \Pi, S)$ denote the spectral radius of the simple random walk on $\Xi \backslash \Pi$ generated by S; and $\alpha = \alpha(\Xi; \Pi, S)$ denote the relative growth of Ξ in Π. Then we have

$$\nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|}\left(\frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{if } \alpha \le \sqrt{|S|-1}. \end{cases} \tag{3.1}$$

Proof. Let $\mathcal{X}$ be the Schreier graph of (Π, S) relative to Ξ defined above. Fix the endpoints $\star = \dagger = \Xi$, the coset of 1, and give $\mathcal{X}$ the length labelling. Let G and F be the circuit and proper circuit series of $\mathcal{X}$. In this setting, expressing $F(t) = \sum f_n t^n$ and $G(t) = \sum g_n t^n$, we see that $|S|\nu$ is the growth rate $\limsup \sqrt[n]{g_n}$ of circuits in $\mathcal{X}$, and α the growth rate $\limsup \sqrt[n]{f_n}$ of proper circuits in $\mathcal{X}$. As both F and G are power series with non-negative coefficients, $1/(|S|\nu)$ is the radius of convergence of G and $1/\alpha$ the radius of convergence of F. Let $d = |S|$ and consider the function

$$(t)\phi = \frac{t}{1+(d-1)t^2} .$$

This function is strictly increasing for $0 \le t < 1/\sqrt{d-1}$, has a maximum at $t = 1/\sqrt{d-1}$ with $(t)\phi = 1/(2\sqrt{d-1})$, and is strictly decreasing for $t > 1/\sqrt{d-1}$.

First we suppose that $\alpha \ge \sqrt{d-1}$, so ϕ is monotonously increasing on $[0, 1/\alpha]$. We set $u = 1$ in (2.2) and note that, for $t < 1$, it says that F has a singularity at t if and only if G has a singularity at $(t)\phi$. Now as $1/\alpha$ is the singularity of F closest to 0, we conclude by monotonicity of ϕ that the singularity of G closest to 0 is at $(1/\alpha)\phi$; thus

$$\frac{1}{d\nu} = \frac{1/\alpha}{1+(d-1)/\alpha^2} = (1/\alpha)\phi .$$

Suppose now that $\alpha < \sqrt{d-1}$. If $d\nu < 2\sqrt{d-1}$, the right-hand side of (2.2) would be bounded for all $t \in \mathbf{R}$ while the left-hand side diverges at $t = 1$. If $d\nu > 2\sqrt{d-1}$, there would be a $t \in [0, 1/\sqrt{d-1}[$ with $(t)\phi = d\nu$; and F would have a singularity at $t < 1/\alpha$. The only case left is $d\nu = 2\sqrt{d-1}$. $\square$

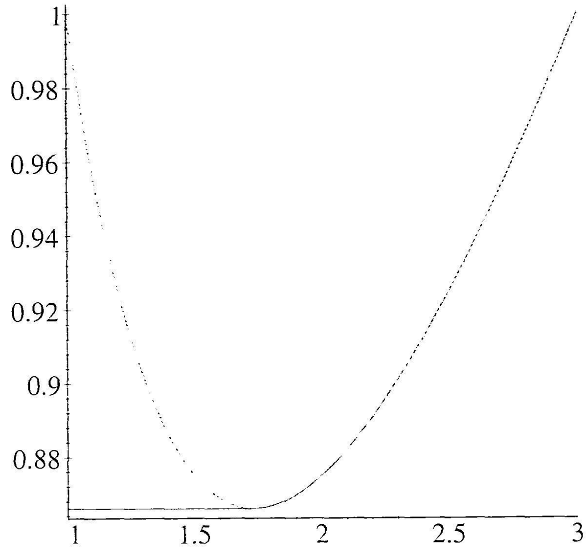

FIGURE 1

The function $\alpha \mapsto \nu$ relating cogrowth and spectral radius (for $d = 4$)

COROLLARY 3.3 (Grigorchuk [Gri78b]). *Let Γ be a group generated by a symmetric finite set S, let ν denote the spectral radius of the simple random walk on Γ, and let α denote the cogrowth of* (Γ, S). *Then*

$$\nu = \begin{cases} \frac{\sqrt{|S|-1}}{|S|}\left(\frac{\alpha}{\sqrt{|S|-1}} + \frac{\sqrt{|S|-1}}{\alpha}\right) & \text{if } \alpha > \sqrt{|S|-1}, \\ \frac{2\sqrt{|S|-1}}{|S|} & \text{else}. \end{cases} \tag{3.2}$$

A variety of proofs exist for this result: the original [Gri78b] by Grigorchuk, one by Cohen [Coh82], an extension by Northshield to regular graphs [Nor92], a short proof by Szwarc [Szw89] using operator theory, one by Woess [Woe94], etc.

Proof. Present Γ as Π/Ξ, with Π a quasi-free group and Ξ the normal subgroup of Π generated by the relators in Γ, and apply Corollary 3.2. □

We note in passing that if $\alpha < \sqrt{|S|-1}$, then necessarily $\alpha = 0$. Equivalently, we will show that if $\alpha < \sqrt{|S|-1}$, then $\Xi = 1$, so the Cayley graph $\mathcal{X}$ is a tree. Indeed, suppose $\mathcal{X}$ is not a tree, so it contains a circuit λ at $\star$. As $\mathcal{X}$ is transitive, there is a translate of λ at every vertex, which we will still write λ. There are at least $|S|(|S|-1)^{t-2}(|S|-2)$ paths p of length t in $\mathcal{X}$ starting at $\star$ such that the circuit $p\lambda\overline{p}$ is proper; thus

$$\alpha \geq \limsup_{t\to\infty} \sqrt[2t+|\lambda|]{|S|(|S|-1)^{t-2}(|S|-2)} = \sqrt{|S|-1}\,.$$

In fact it is known that $\alpha > \sqrt{|S|-1}$; see [Pas93].

3.2 The series F and G on their circle of convergence

In this subsection we study the singularities the series F and G may have on their circle of convergence. The smallest positive real singularity has a special importance:

DEFINITION 3.4. For a series $f(t)$ with positive coefficients, let $\rho(f)$ denote its radius of convergence. Then f is *$\rho(f)$-recurrent* if

$$\lim_{t \to \rho(f)} f(t) = \infty .$$

Otherwise, it is *$\rho(f)$-transient.*

As typical examples, $1/(\rho - t)$ is ρ-recurrent, as are all rational series; $\sqrt{\rho - t}$ is ρ-transient, while $1/\sqrt{\rho - t}$ is not.

To study the singularities of F or G, we may suppose that $\star = \dagger$; indeed in was shown in [Kes59] and [Woe83, Lemma 1] that the singularities of F and G do not depend on the choice of $\star$ and $\dagger$. We make that assumption for the remainder of the subsection. We will also suppose throughout that $\mathcal{X}$ is d-regular, that the radius of convergence of F is $1/\alpha$ and the radius of convergence of G is $1/(d\nu) = 1/\beta$.

DEFINITION 3.5. Let $\mathcal{X}$ be a connected graph. A *proper cycle* in $\mathcal{X}$ is a proper circuit $(\pi_1, \dots, \pi_n)$ such that $\overline{\pi_1} \neq \pi_n$. The *proper period* p and *strong proper period* p_s are defined as follows:

$$p = \gcd\{n \mid \text{there exists a proper cycle } \pi \text{ in } \mathcal{X} \text{ with } |\pi| = n\},$$

$$p_s = \gcd\{n \mid \forall x \in V(\mathcal{X}) \text{ there exists a proper cycle } \pi \text{ in } \mathcal{B}(x,n) \text{ with } |\pi| = n\},$$

where by convention the gcd of the empty set is ∞. The graph $\mathcal{X}$ is *strongly properly periodic* if $p = p_s$.

The *period* q and *strong period* q_s of $\mathcal{X}$ are defined analogously with 'proper cycle' replaced by 'circuit'. $\mathcal{X}$ is *strongly periodic* if $q = q_s$.

THEOREM 3.6 (Cartwright [Car92]). *Let $\mathcal{X}$ have proper period p and strong proper period p_s. Then the singularities of F on its circle of convergence are among the*

$$\frac{e^{2i\pi k/p_s}}{\alpha}, \quad k = 1, \dots, p_s .$$

If moreover $\mathcal{X}$ is strongly properly periodic, the singularities of F on its circle of convergence are precisely these numbers.

Let $\mathcal{X}$ have period q and strong period q_s. Then the singularities of G on its circle of convergence are among the

$$\frac{e^{2i\pi k/q_s}}{\beta}, \quad k = 1, \dots, q_s .$$

If moreover $\mathcal{X}$ is strongly periodic, the singularities of G on its circle of convergence are precisely these numbers.

If $\mathcal{X}$ is connected and non-trivial, there is a path of even length at every vertex (a sequence of bumps, for instance). All graphs are then either 2-periodic (if they are bipartite) or 1-periodic. If there is a constant N such that for all $x \in V(\mathcal{X})$ there is at x a circuit of odd length bounded by N, then $\mathcal{X}$ is strongly 1-periodic; otherwise $\mathcal{X}$ is strongly 2-periodic. The singularities of G on its circle of convergence are then at $1/\beta$, and also at $-1/\beta$ if $\mathcal{X}$ is strongly periodic with period 2.

If $\mathcal{X}$ is not strongly periodic, there may be one or two singularities on G's circle of convergence; consider for instance the 4-regular tree, and at a vertex $\star$ delete two or three edges replacing them by loops. The resulting graphs $\mathcal{X}_2$ and $\mathcal{X}_3$ are still 4-regular and their circuit series, as computed using (7.2), are respectively

$$(3.3) \qquad \begin{aligned} G_2(t) &= \frac{3}{2 - 6t + \sqrt{1 - 12t^2}}, \\ G_3(t) &= \frac{6}{5 - 18t + \sqrt{1 - 12t^2}}. \end{aligned}$$

G_2 has singularities at $\pm 1/\sqrt{12}$ on its circle of convergence, while G_3 has only $2/7$ as singularity on its circle of convergence.

Following the proof of Corollary 3.2 above, we see that if $\beta < d$ the singularities of F on its circle of convergence are in bijection with those of G, so are at $1/\alpha$ and possibly $-1/\alpha$, if $\mathcal{X}$ is strongly two-periodic. If $\beta = d$, though, $\mathcal{X}$ can have any strong proper period; consider for example the cycles on length k studied in Section 7.2: they are strongly properly k-periodic.

The forthcoming simple result shows how $\mathcal{X}$ can be approximated by finite subgraphs.

LEMMA 3.7. *Let $\mathcal{X}$ be a graph and x, y two vertices in $\mathcal{X}$. Let $\mathfrak{G}_{x,y}$ and $\mathfrak{F}_{x,y}$ be the path series and enriched path series respectively from x to y in $\mathcal{X}$, and let $\mathfrak{G}^n_{x,y}$ and $\mathfrak{F}^n_{x,y}$ be the path series and enriched path series respectively from x to y in the ball $\mathcal{B}(x,n)$ (they are 0 if $y \notin \mathcal{B}(x,n)$). Then*

$$\lim_{n\to\infty} \mathfrak{G}^n_{x,y} = \mathfrak{G}_{x,y}, \qquad \lim_{n\to\infty} \mathfrak{F}^n_{x,y} = \mathfrak{F}_{x,y}.$$

Proof. Recall that $\lim \mathfrak{G}^n_{x,y} = \mathfrak{G}_{x,y}$ means that both terms are sums of paths, say A_n and A, such that the minimal length of paths in the symmetric difference $A_n \triangle A$ tends to infinity. Now the difference between $\mathfrak{G}^n_{x,y}$ and $\mathfrak{G}_{x,y}$ consists only of paths in $\mathcal{X}$ that exit $\mathcal{B}(x,n)$, and thus have length at least $2n - \delta(x,y) \to \infty$. The same argument holds for $\mathfrak{F}$. □

DEFINITION 3.8. The graph $\mathcal{X}$ is *quasi-transitive* if $\mathrm{Aut}(\mathcal{X})$ acts with finitely many orbits.

LEMMA 3.9. *Let $\mathcal{X}$ be a regular quasi-transitive connected graph with distinguished vertex $\star$, and let f_n and g_n denote respectively the number of proper circuits and circuits at $\star$ of length n. Then*

$$\limsup_{n\to\infty} g_n/\beta^n = \limsup_{n\to\infty} f_n/\alpha^n = \begin{cases} 1/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has odd circuits;} \\ 2/|\mathcal{X}| & \text{if } \mathcal{X} \text{ is finite and has only even circuits;} \\ 0 & \text{if } \mathcal{X} \text{ is infinite.} \end{cases}$$

Proof. If $\mathcal{X}$ is finite, then $\beta = d$, the degree of $\mathcal{X}$; after a large even number of steps, a random walk starting at $\star$ will be uniformly distributed over $\mathcal{X}$ (or over the vertices at even distance of $\star$, in case all circuits have even length). A long enough walk then has probability $1/|\mathcal{X}|$ (or $2/|\mathcal{X}|$ if all circuits have even length) of being a circuit.

If $\mathcal{X}$ is infinite, we consider two cases. If $G(1/\beta) < \infty$, i.e. G is $1/\beta$-transient, the general term g_n/β^n of the series $G(1/\beta)$ tends to 0. If G is $1/\beta$-recurrent, then, as $\mathcal{X}$ is quasi-transitive, $\beta = d$ by [Woe98, Theorem 7.7]. We then approximate $\mathcal{X}$ by the sequence of its balls of radius R, by Lemma 3.7:

$$\lim_{n\to\infty} \frac{g_n}{\beta^n} = \lim_{R,n\to\infty} \frac{g_{R,n}}{d^n} = \lim_{R\to\infty} \frac{(1 \text{ or } 2)}{|\mathcal{B}(\star, R)|} = 0,$$

where we expand the circuit series of $\mathcal{B}(\star, R)$ as $\sum g_{R,n} t^n$.

The same proof holds for the f_n. Its particular case where $\mathcal{X}$ is a Cayley graph appears in [Woe83]. □

Note that if $\mathcal{X}$ is not quasi-transitive, a somewhat weaker result holds [Kit98, §7.1]: if $\mathcal{X}$ is transient or null-recurrent then the common limsup is 0. If $\mathcal{X}$ is positive-recurrent then the limsups are normalized coefficients of $\mathcal{X}$'s Perron-Frobenius eigenvector. Lemma 3.9 is not true for arbitrary d-regular graphs: consider for instance the graph $\mathcal{X}_3$ described above. Its circuit series G_3, given in (3.3), has radius of convergence $1/\beta = 2/7$, and one easily checks that all its coefficients g_n satisfy $g_n/\beta^n \geq 1/2$.

We obtain the following characterization of rational series:

THEOREM 3.10. *For regular quasi-transitive connected graphs $\mathcal{X}$, the following are equivalent:*

1. *$\mathcal{X}$ is finite;*
2. *$G(t)$ is a rational function of t;*
3. *$F(t)$ is a rational function of t, and $\mathcal{X}$ is not an infinite tree.*

Proof. By Corollary 2.7, Statement 1 implies the other two. By Corollary 2.6, and a computation on trees done in Section 7.3 to deal with the case $F(t) = 1$, Statement 2 implies 3. It remains to show that Statement 3 implies 1.

Assume that $F(t) = \sum f_n t^n$ is rational, not equal to 1. As the f_n are positive, F has a pole, of multiplicity m, at $1/\alpha$. There is then a constant $a > 0$ such that $f_n > a\binom{n}{m-1}\alpha^n$ for infinitely many values of n [GKP94, page 341]. It follows by Lemma 3.9 that $m = 1$ and the graph $\mathcal{X}$ is finite, of cardinality at most $1/a$. $\square$

It is not known whether the same holds for regular, or even arbitrary connected graphs. Certainly an altogether different proof would be needed.

3.3 APPLICATION TO LANGUAGES

Let S be a finite set of cardinality d and let $\bar{\cdot}$ be an involution on S. A *word* is an element w of the free monoid S^*. A *language* is a set L of words. The language L is called *saturated* if for any $u, v \in S^*$ and $s \in S$ we have

$$uv \in L \Longleftrightarrow us\bar{s}v \in L;$$

that is to say, L is stable under insertion and deletion of subwords of the form $s\bar{s}$. The language L is called *desiccated* if no word in L contains a subword of the form $s\bar{s}$. Given a language L we may naturally construct its *saturation*

$\langle L \rangle$, the smallest saturated language containing L, and its *desiccation* $\widehat{L}$, the largest desiccated language contained in L.

Let Σ be the monoid defined by generators S and relations $s\bar{s} = 1$ for all $s \in S$:

$$\Sigma = \langle S \mid s\bar{s} = 1 \quad \forall s \in S \rangle . \tag{3.4}$$

This is a free product of free groups and order-two groups; if $\bar{\cdot}$ is fixed-point-free, Σ is a free group. Write ϕ for the canonical projection from S^* to Σ. Let $\mathbf{k} = \mathbf{Z}[\Sigma]$ be its monoid ring. Then given a language $L \subset S^*$ we may define its *growth series* $\Theta(L)$ as

$$\Theta(L) = \sum_{w \in L} w^\phi t^{|w|} \in \mathbf{k}[[t]] .$$

This notion of growth series with coefficients was introduced by Fabrice Liardet in his doctoral thesis [Lia96], where he studied *complete growth functions* of groups.

THEOREM 3.11. *For any language L there holds*

$$\frac{\Theta(\widehat{L})(t)}{1-t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} , \tag{3.5}$$

where $d = |S|$.

Proof. For any language there exists a unique minimal (possibly infinite) automaton recognising it ([Eil74, § III.5] is a good reference). Let $\mathcal{X}$ be the minimal automaton recognising $\langle L \rangle$. Recall that this is a graph with an initial vertex $\star$, a set of terminal vertices T and a labelling $\ell' \colon E(\mathcal{X}) \to S$ of the graph's edges such that the number of paths labelled w, starting at $\star$ and ending at a $\tau \in T$ is 1 if $w \in L$ and 0 otherwise. Extend the labelling ℓ' to a labelling $\ell \colon E(\mathcal{X}) \to \mathbf{k}[[t]]$ by

$$e^\ell = t \cdot (e^{\ell'})^\phi .$$

Because $\langle L \rangle$ is saturated, and $\mathcal{X}$ is minimal, $(\bar{e})^\ell = \overline{e^\ell}$; then $\widehat{L}$ is the set of labels on proper paths from $\star$ to some $\tau \in T$. Choosing in turn all $\tau \in T$ as $\dagger$, we obtain growth series F_τ, G_τ counting the formal sum of paths and proper paths from $\star$ to τ. It then suffices to write

$$\frac{\Theta(\widehat{L})(t)}{1-t^2} = \frac{\sum_{\tau \in T} F_\tau(t)}{1-t^2} = \frac{\sum_{\tau \in T} G_\tau\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} = \frac{\Theta(\langle L \rangle)\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2} . \qquad \square$$

The following result is well-known:

THEOREM 3.12 (Müller & Schupp [MS81, MS83]). *Let Γ be a finitely generated group, presented as a quotient Σ/Ξ with Σ as in (3.4). Then $\Theta(\Xi)$ is an algebraic series (i.e. satisfies a polynomial equation over $\mathbf{k}[t]$) if and only if Σ/Ξ is virtually free (i.e. has a normal subgroup of finite index that is free).*

It is not known whether there exists a non-virtually-free quasi-transitive graph whose circuit series (as defined in Corollary 2.6) is algebraic.

4. FIRST PROOF OF THEOREM 2.4

We now prove Theorem 2.4 using linear algebra. We first assume the graph has a finite number of vertices, for the computations refer to $\mathbf{k}$-matrices and $\mathbf{k}[[u]]$-matrices indexed by the graph's vertices. This proof is hinted at in Godsil's book as an exercise [God93, page 72]; it was also suggested to the author by Gilles Robert.

For all pairs of vertices $x, y \in V(\mathcal{X})$ let

$$\mathfrak{G}_{x,y}(\ell) = \sum_{\pi \in [x,y]} \pi^{\ell}, \qquad \mathfrak{F}_{x,y}(\ell) = \sum_{\pi \in [x,y]} u^{\mathrm{bc}(\pi)} \pi^{\ell}$$

be the path and enriched path series from x to y; for ease of notation we will leave out the labelling ℓ if it is obvious from the context. Let $\delta_{x,y}$ denote the Kronecker delta, equal to 1 if $x = y$ and 0 otherwise. For any $v \in \mathbf{k}$, let $[v]_x^y$ denote the $V(\mathcal{X}) \times V(\mathcal{X})$ matrix with zeros everywhere except at (x, y), where it has value v. Then

$$\mathfrak{G}_{x,y} = \delta_{x,y} + \sum_{e \in E(\mathcal{X}) : e^{\alpha} = x} e^{\ell} \mathfrak{G}_{e^{\omega},y}$$

so that if

$$A = \sum_{e \in E(\mathcal{X})} [e^{\ell}]_{e^{\alpha}}^{e^{\omega}}$$

be the adjacency matrix of $\mathcal{X}$, with labellings, then we have

$$(\mathfrak{G}_{x,y})_{x,y \in V(\mathcal{X})} = \frac{1}{1 - A},$$

an equation holding between $V(\mathcal{X}) \times V(\mathcal{X})$ matrices over $\mathbf{k}$.