3. Hyperspaces and dual trees

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

3. Hyperspaces and dual trees

In this section, we assume that X is an n-dimensional simply connected cubical chamber complex of nonpositive curvature, endowed with the cubical metric.

Hyperspaces

Let P be a k-cell in X, $1 \le k \le n$. Any subset of P of the form $\{\frac{1}{2}\} \times [0,1]^{k-1}$, for any isometric identification of P with $[0,1]^k$, is called a *wall* in P. If Q is a j-cell of X contained in P, $1 \le j < k$, and W is a wall in Q, then there is precisely one wall V in P such that $V \cap P = W$. Such a wall V is perpendicular to Q in P. In particular, if Q is an edge, there is precisely one wall V in P such that $V \cap P$ is the midpoint of Q and V is perpendicular to Q.

LEMMA 3.1. *Let P be a k-cell in X and W a wall in P. Then* $\operatorname{res} P$ *is isometric to* $\operatorname{res} W \times [0,1]$, *where* $\operatorname{res} W := \bigcup V$ *and the union is over the walls V in cells $Q \in \operatorname{res} P$ such that $V \cap P = W$.* $\square$

LEMMA 3.2. *A wall W in a cell P extends uniquely to a minimal connected subspace $\Sigma = \Sigma_W \subset X$ such that*

(1) *Σ is a union of walls;*

(2) $\operatorname{res} V \subset \Sigma$ *for any wall $V \subset \Sigma$.*

Moreover,

(3) *if Σ intersects a cell P then* $\Sigma \cap \operatorname{res} P = \operatorname{res} W$ *for some wall W of P;*

(4) *Σ is locally (and hence globally) convex; and*

(5) *$X \setminus \Sigma$ consists of two convex connected components.*

Proof. Existence and uniqueness of a connected subspace satisfying Properties (1) and (2) is clear from what was said before. Property (3) follows from the observation that otherwise it would be possible to find in X a nontrivial geodesic (contained in Σ) with the same initial and final point (belonging to the "selfintersection locus" of Σ). Property (4) is then an immediate consequence of (3), Lemma 3.1 and Theorem 1.4(2). Property (5) follows from the contractibility of X: we have to exclude the existence of a closed curve in X that crosses Σ once. Now such a closed curve can be contracted to a constant curve and a contraction can be put into general position with respect to Σ. Then the number of transversal intersections with

Σ does not change mod 2. Since this number is 0 for the final constant curve, it cannot be 1 for the initial curve. The two resulting components of $X \setminus \Sigma$ are (globally) convex since, by (3) and Lemma 3.1 they are clearly locally convex. $\square$

We call the subspaces Σ as above *hyperspaces* in X.

DUAL TREES

From now on we assume that X is a simply connected foldable cubical chamber complex of nonpositive curvature. Fix a folding $F: X \to C$ of X onto an n-dimensional cube C, $n = \dim X$. Label the walls in C by the numbers $1, \dots, n$ and the panels of C by the label of the corresponding parallel wall. Lift these labellings by F to the walls and panels in the chambers of X. Each hyperspace Σ in X is a union of walls of chambers of X, and the labels of the walls in Σ are the same. Thus we also obtain a labelling of the hyperspaces. Two different hyperspaces with the same label are disjoint.

Denote by Λ_i the union of the walls with label i in the chambers of X. Then Λ_i is the union of the hyperspaces labelled i. Moreover, the intersection of the boundaries of two different connected components of $X \setminus \Lambda_i$ is either empty or a hyperspace with label i. Therefore we can define a graph Λ_i^* as follows: the vertices of Λ_i^* correspond to the connected components of $X \setminus \Lambda_i$; two vertices are connected by an edge if the corresponding components are adjacent along a hyperspace with label i. Observe that Λ_i^* is a tree since the complement of any of its edges is disconnected by the separating property of hyperspaces, see Lemma 3.2(5). We call Λ_i^* the *dual tree* to the system of hyperspaces with label i. Note that in general Λ_i^* may not be locally finite, even if the initial complex X is. We endow Λ_i^* with the length metric d_i^* such that each edge has length 1.

Note that the panels of X with label i do not belong to the set Λ_i, $1 \le i \le n$. Thus we can define maps $r_i: X \to \Lambda_i^*$ as follows: a panel of X is mapped by r_i to the vertex of Λ_i^* representing the component in $X \setminus \Lambda_i$ to which it belongs. This extends uniquely to all chambers of X so that a chamber P is mapped by r_i onto the edge in Λ_i^* representing the hyperspace in X containing the wall of P labelled i and such that r_i is isometric in the direction perpendicular to the wall with label i.

The same argument as in the proof of Lemma 3.2(4) shows that the preimage $r_i^{-1}(p)$ of any point $p \in \Lambda_i^*$ distinct from a vertex is a convex subset of X. Moreover, if p is a vertex of Λ_i^*, then the convexity of the subcomplex $r_i^{-1}(p) \subset X$ follows from foldability of links of X at vertices in view of the following characterisation (see e.g. Lemma 1.7.1 in [DJS]): a connected subcomplex K in a simply connected nonpositively curved cubical complex L is convex if and only if for each vertex v of K the link K_v is a full subcomplex of the link L_v (which means that a simplex of L_v belongs to K_v whenever its vertices belong to K_v). The above properties imply that if $\sigma\colon I \to X$ is a geodesic, then $r_i \circ \sigma$ is (weakly) monotonic: $r_i \circ \sigma$ never turns. Furthermore, if σ is not constant, then for each $t \in I$ there are $i, j \in \{1, \dots, n\}$ such that $r_i \circ \sigma$ is injective on $(t-\varepsilon, t] \cap I$ and $r_j \circ \sigma$ is injective on $[t, t+\varepsilon) \cap I$.

EMBEDDING INTO A PRODUCT OF TREES

Consider the map $r\colon X \to \prod_{i=1}^n \Lambda_i^*$ defined by $r(x) = (r_1(x), \dots, r_n(x))$. Clearly r is a nondegenerate combinatorial map of cubical complexes, that is, it is isometric on each cell of X. By what we just said about the image of geodesics under the maps r_i, it follows immediately that r is injective. We call r the *canonical embedding* of X into the product of trees $\prod_{i=1}^n \Lambda_i^*$.

Recall that d_i^* is the natural metric in Λ_i^*. Define two metrics $d_{(1)}$ and $d_{(2)}$ on the product $\prod_{i=1}^n \Lambda_i^*$ by

$$d_{(1)} = \sum_{i=1}^n d_i^* \quad \text{and} \quad d_{(2)} = \Big(\sum_{i=1}^n (d_i^*)^2\Big)^{\frac{1}{2}}. \tag{3.3}$$

It is easy to see that $d_{(2)} \le d_{(1)} \le \sqrt{n} \cdot d_{(2)}$, and hence the two metrics are Lipschitz equivalent. Moreover, we have

PROPOSITION 3.4. *The map r is a biLipschitz embedding. More precisely, if x and y are points in X, then*

$$d_{(2)}(r(x), r(y)) \le d(x, y) \le d_{(1)}(r(x), r(y)).$$

where d denotes the cubical metric on X.

Proof. The first inequality follows from the fact that r restricted to any chamber of X is an isometry. The second inequality is obviously true for x and y belonging to the same chamber of X. It extends to arbitrary x and y since for each geodesic σ in X, $r_i \circ \sigma$ is monotonic and hence, up to parameter, a geodesic in Λ_i^*. $\square$

EQUIVARIANCE PROPERTIES OF THE CANONICAL EMBEDDING

It follows from gallery connectedness of X that the folding map $F: X \to C$ is unique up to an automorphism of C, so that a group Γ acting by automorphisms on X has a well defined homomorphism into the group $\mathrm{Aut}(C)$ of all automorphisms of C. The kernel Γ' of this homomorphism is a finite index subgroup in Γ, it preserves all the sets Λ_i and hence acts by automorphisms on the dual trees Λ_i^*.

From now on, we assume that Γ preserves the folding of X and hence the labelling of the walls. Then Γ acts on the dual trees Λ_i^* and the maps r_i are equivariant with respect to these actions. Therefore the canonical embedding r is equivariant with respect to the diagonal action of Γ on the product $\prod_{i=1}^n \Lambda_i^*$. This completes the proof of the first assertion of Theorem 1 in the introduction.

Since r is equivariant, it follows that $\mathrm{Stab}(\Gamma, x) \subset \mathrm{Stab}(\Gamma, r(x))$ for each $x \in X$, where $\mathrm{Stab}(G, p)$ denotes the stabilizer of a point p with respect to a transformation group G.

PROPOSITION 3.5. *For each $p \in \prod_{i=1}^n \Lambda_i^*$, there is a point $x_p \in X$ such that $Stab(\Gamma, p) \subset Stab(\Gamma, x_p)$. In particular, if Γ does not have a fixed point in X, then Γ acts without a fixed point on at least one of the trees Λ_i^*.*

Proof. If p is in the image of r, then the assertion follows from the injectivity of r. If not, let δ be the distance of p to the image of r with respect to the metric $d_{(2)}$. Take the ball $B(p, 2\delta)$ of radius 2δ about p in $(\prod_{i=1}^n \Lambda_i^*, d_{(2)})$. The preimage $r^{-1}(B(p, 2\delta))$ is then a bounded nonempty subset of X by Proposition 3.4. Let x_p be its circumcenter, i.e. the center of the unique ball with smallest radius containig this subset, see [Ba, p. 26]. Since Γ acts by isometries with respect to $d_{(2)}$, $B(p, 2\delta)$ is fixed by each automorphism in $\mathrm{Stab}(\Gamma, p)$. Since r is equivariant and Γ acts by isometries on X, each such automorphism fixes $r^{-1}(B(p, 2\delta))$ and hence x_p. □

Our next proposition is a special case of a more general result of M. Bridson [B2]. Together with Proposition 3.5, it completes the proof of Theorem 1 of the introduction. For the convenience of the reader we include a short proof adapted to our case of folded cubical complexes.

PROPOSITION 3.6. *Let X be a simply connected, folded cubical chamber complex of nonpositive curvature. Then any automorphism of X is semisimple, i.e. elliptic or axial.*

Proof. Let φ be an automorphism of X. If φ fixes a point p of Λ_i^*, then p can be chosen as a vertex or a midpoint of an edge. If p is a vertex, then the preimage X' of p under r_i is a closed and convex subcomplex of X. If p is the midpoint of an edge, X' is a hyperspace and as a union of walls, carries a natural cubical structure. In either case, X' is a closed, convex and φ-invariant subset of X, and therefore φ is semisimple if and only if the restriction $\varphi|_{X'}$ is semisimple. Since moreover X' is a simply connected folded cubical chamber complex of nonpositive curvature and of dimension lower than X, we can assume by induction on $\dim X$ that the action of φ on all the trees Λ_i^* is axial.

Let a_i be an axis of φ in Λ_i^* (unique up to parameter). Let $X_i = r_i^{-1}(a_i)$. Since r_i is surjective, X_i is non-empty. Furthermore, X_i is a closed, convex and φ-invariant subcomplex of X.

Set $Y_1 := X_1$. The image of Y_1 under r_2 is path connected and φ-invariant, hence contains a_2. Let $Y_2 = Y_1 \cap X_2$. Then Y_2 is non-empty, closed, convex and φ-invariant. By induction we get that $Y = X_1 \cap \ldots \cap X_n$ is a non-empty, closed, convex and φ-invariant subcomplex of X. It is then sufficient to prove semisimplicity for the restriction $\varphi|_Y$. Note that $Y = r^{-1}(F)$, where $F \cong \mathbf{R}^n$ is the flat

$$F = \{(a_1(t_1), \ldots, a_n(t_n)) \mid t_i \in \mathbf{R}\}$$

in the product of trees. Now φ operates as a translation on F, hence the displacement of φ on F is constant, say $= \delta$. Since r is injective, we can consider Y as a closed subcomplex of F, namely a union of chambers. The metric on Y is the induced path metric. It follows easily that there are only finitely many possible values for the distance in Y from a point x to its image φx, if the location of x in its chamber is given. $\square$

4. Nonexistence of free subgroups

In this section we discuss the proof of Theorem 2 of the introduction. We assume throughout this section that X is a simply connected folded cubical chamber complex of nonpositive curvature and that $\Gamma \subset \mathrm{Aut}(X)$ is a group that preserves the folding of X (this can be always assumed by passing to a finite index normal subgroup if necessary) and does not contain a free nonabelian subgroup acting freely on X. By equivariance of the maps r_i, the same holds for the actions of Γ on the trees Λ_i^*. Up to a subgroup of index two, there are three possibilities for each particular i [PV]: