2. The generalized Følner condition

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.04.2024**

Let $||P||$ be the operator norm of P acting on $l^2(\Gamma)$. In [8] Kesten proved:

THEOREM 1 (Kesten). *The following conditions are equivalent:*

(1) $||P|| = 1$.

(2) *The group* Γ *is amenable, i.e. there exists a sequence* $\{A_n\}_{n=1}^{\infty}$ *of finite subsets of* Γ *satisfying the Følner condition.*

In the next section we will prove a generalization of this result (Theorems 2 and 3), showing that equalities of the form $||P|| = \lambda$, with $0 < \lambda \leq 1$, are equivalent to appropriate Følner-like conditions. Section 3 is devoted to some remarks concerning this generalization. In Section 4 we use the generalized Følner condition to compute the norms of some random walk operators and in Section 5, using the same ideas, we obtain some lower bounds for the random walk operators on graphs.

After completion of this work, we learned that some versions of a generalized Følner condition were obtained recently by S. Popa [12].

ACKNOWLEDGEMENTS. I would like to express my gratitude to A. Hulanicki and to L. Saloff-Coste for several interesting discussions and remarks on the paper, and for suggesting the example in Section 4.4. I also wish to thank P. de la Harpe and the referee for their several valuable comments on this paper. This work was done with the support of the Swiss National Science Foundation.

2. THE GENERALIZED FØLNER CONDITION

Let us consider a measurable space $(X, \mathcal{F})$. On this space we consider a *Markov transition kernel* $P(\cdot, \cdot)$, i.e. for any $x \in X$, $P(x, \cdot)$ is a probability measure on $(X, \mathcal{F})$ and $P(\cdot, A)$ is a measurable function on $(X, \mathcal{F})$ for every $A \in \mathcal{F}$.

Let μ be a σ-finite measure on the space $(X, \mathcal{F})$. For any measurable subset $A \subset X$ we define its measure $|A|$ and the measure $|\partial A|$ of its boundary ∂A as follows:

$$|A| = \mu(A),$$
$$|\partial A| = \int_{\{x \in A\}} \int_{\{y \in A^c\}} P(x, dy) d\mu(x).$$

We will suppose that the measure

$$dm(x,y) = d\mu(x)P(x,dy) \tag{1}$$

is symmetric on $X \times X$. Let P be the *Markov operator* acting on $L^2(X,\mu)$ as

$$Pf(x) = \int_{\{y \in X\}} f(y)P(x,dy)\,.$$

The above equation defines also an operator on the space of positive measurable functions on X.

When condition (1) is satisfied we say that P is *reversible* with respect to μ. The Markov operator P is a self-adjoint operator on $L^2(X,\mu)$ if and only if P is reversible with respect to μ.

We denote by $\langle \cdot, \cdot \rangle_{L^2(X,\mu)}$ the scalar product on $L^2(X,\mu)$ and by $\|P\|$ the norm of P acting on $L^2(X,\mu)$.

For a real-valued measurable function f and for a measurable subset $A \subset X$ let us define a relative measure $|A|_{f^2}$ and a relative measure of its boundary $|\partial A|_{f^2}$:

$$|A|_{f^2} = \int_{\{x \in A\}} f^2 d\mu(x)\,,$$

$$|\partial A|_{f^2} = \int_{\{x \in A\}} \int_{\{y \in A^c\}} f(x)f(y)P(x,dy)d\mu(x)\,.$$

THEOREM 2. *Let P be a Markov operator on a measurable space $(X,\mathcal{F})$, which is reversible with respect to a measure μ. Let f be a positive eigenfunction of P with a positive eigenvalue λ. Then the following conditions are equivalent:*

(1) *There is a constant $c > 0$ such that for any measurable subset $A \subset X$ of finite measure*

$$|A|_{f^2} \leq c|\partial A|_{f^2}\,,$$

(2) $\|P\| < \lambda$.

In the case where X is a Cayley graph of a group Γ with a finite set of generators $S = S^{-1}$ like in Part 1, one can give the following formulation of the above theorem.

THEOREM 3. *Let f be a positive eigenfunction for the simple random walk operator P on the group Γ generated by a finite symmetric set S, with the eigenvalue λ, i.e.*

$$Pf = \lambda f .$$

The following conditions are equivalent:

(1) $\|P\| = \lambda$.

(2) (Generalized Følner condition) *There exists a sequence $\{A_n\}_{n=1}^{\infty}$ of finite subsets of Γ, such that*

$$\frac{\sum_{\gamma \in \partial A_n} f^2(\gamma)}{\sum_{\gamma \in A_n} f^2(\gamma)} \to_{n \to \infty} 0 .$$

REMARK. In case where $\lambda = 1$ we can take the function f of Theorem 3 to be a constant function. We then obtain Kesten's theorem (Theorem 1).

There are also examples of amenable groups (see [2]) for which there exist eigenfunctions of the simple random walk operator corresponding to the eigenvalue equal to one and which are not constant. The generalized Følner condition applies also to them.

Theorem 2 will be deduced from the following proposition.

PROPOSITION 1 ([7,13]). *Let Q be a Markov operator on $(X, \mathcal{F})$ which is reversible with respect to a measure μ. Assume that there exists a constant $c > 0$ such that for any measurable subset $A \subset X$ of finite measure*

$$|A| \leq c|\partial A| . \tag{2}$$

Then

$$\|Q\|_{L^2(X,\mu) \to L^2(X,\mu)} \leq 1 - \frac{1}{\sqrt{2}c} < 1 .$$

In order to give a clear proof of Proposition 1, we need the following lemma.

LEMMA 1 ([13]). *For a non-negative measurable function f with compact support in X one has*

$$\int_{\{x \in X\}} \int_{\{y \in X\}} |f(x) - f(y)| Q(x, dy) d\mu(x) = 2 \int_0^{\infty} |\partial\{f > t\}| dt .$$

Proof.

$$\int_{\{x\in X\}}\int_{\{y\in X\}}|f(x)-f(y)|Q(x,dy)d\mu(x)$$
$$=2\int_{\{x\in X\}}\int_{\{y\in X;f(x)>f(y)\}}(f(x)-f(y))Q(x,dy)d\mu(x)$$
$$=2\int_{\{x\in X\}}\int_{\{y\in X;f(x)>f(y)\}}\int_0^\infty \mathbf{1}_{[f(y),f(x))}(t)dtQ(x,dy)d\mu(x)$$
$$=2\int_0^\infty\int_{\{x\in X\}}\int_{\{y\in X;f(x)>f(y)\}}\mathbf{1}_{[f(y),f(x))}(t)Q(x,dy)d\mu(x)dt$$
$$=2\int_0^\infty\Big(\int_{\{x\in X;f(x)>t\}}\int_{\{y\in X;f(y)\le t\}}Q(x,dy)d\mu(x)\Big)dt$$
$$=2\int_0^\infty|\partial\{f>t\}|dt\,. \qquad \square$$

Proof of Proposition 1. Let us consider a real-valued measurable function f with compact support in X. The above lemma applied to the function f^2 and the strong isoperimetry condition (2) gives:

$$\int_{\{x\in X\}}\int_{\{y\in X\}}\left|f^2(x)-f^2(y)\right|Q(x,dy)d\mu(x)=2\int_0^\infty\left|\partial\{f^2>t\}\right|dt$$
$$\ge\frac{2}{c}\int_0^\infty\left|\{f^2>t\}\right|dt=\frac{2}{c}\int_X f^2(x)d\mu(x)\,.$$

On the other hand

$$\int_{\{x\in X\}}\int_{\{y\in X\}}\left|f^2(x)-f^2(y)\right|Q(x,dy)d\mu(x)$$
$$\le\int_{\{x\in X\}}\int_{\{y\in X\}}|f(x)-f(y)|(|f(x)|+|f(y)|)Q(x,dy)d\mu(x)$$
$$=2\int_{\{x\in X\}}\int_{\{y\in X\}}|f(x)-f(y)||f(x)|Q(x,dy)d\mu(x)$$
$$=2\int_{\{x\in X\}}\Big(\int_{\{y\in X\}}|f(x)-f(y)|Q(x,dy)\Big)|f(x)|d\mu(x)$$
$$\le2\int_{\{x\in X\}}\Big(\int_{\{y\in X\}}|f(x)-f(y)|^2Q(x,dy)\Big)^{\frac12}|f(x)|d\mu(x)$$
$$\le2\Big(\int_{\{x\in X\}}\int_{\{y\in X\}}|f(x)-f(y)|^2Q(x,dy)d\mu(x)\Big)^{\frac12}\Big(\int_{\{x\in X\}}|f(x)|^2d\mu(x)\Big)^{\frac12}$$
$$=2\sqrt{2}\langle(I-Q)f,f\rangle^{\frac12}_{L^2(X,\mu)}\|f\|_{L^2(X,\mu)}\,.$$

Hence

$$\langle (I-Q)f,f\rangle_{L^2(X,\mu)} \geq \frac{1}{\sqrt{2}c}\|f\|_{L^2(X,\mu)}\,.$$

As Q is a self-adjoint operator, this is equivalent to

$$\|Q\|_{L^2(X,\mu)\to L^2(X,\mu)} \leq 1-\frac{1}{\sqrt{2}c} < 1\,. \qquad \square$$

Let P be a Markov operator, reversible with respect to the measure μ. Let f be a positive eigenfunction of P for the eigenvalue λ.

LEMMA 2. *The operator defined by the kernel*

$$Q(x,dy)=\lambda^{-1}f(x)^{-1}P(x,dy)f(y)$$

is a Markov operator and is reversible with respect to the measure $f^2\mu$.

Proof. The kernel $Q(x,dy)$ is a Markov transition kernel, because:

$$\int_{\{y\in X\}} Q(x,dy)=\lambda^{-1}f(x)^{-1}\int_{\{y\in X\}} P(x,dy)f(y)=\lambda^{-1}f(x)^{-1}\lambda f(x)=1\,.$$

In order to prove reversibility of Q, we have to prove that the measure

$$dm'(x,y)=f^2(x)d\mu(x)Q(x,dy)$$

is symmetric on $X\times X$, knowing that the measure

$$dm(x,y)=d\mu(x)P(x,dy)$$

is symmetric on $X\times X$.

This follows from the following equalities, where B is a measurable subset of $X\times X$:

$$\begin{aligned}\int_B dm'(x,y) &= \int_B f^2(x)d\mu(x)Q(x,dy)\\ &= \lambda^{-1}\int_B f(x)d\mu(x)P(x,dy)f(y)\\ &= \lambda^{-1}\int_B f(x)f(y)dm(x,y)\\ &= \lambda^{-1}\int_B f(x)f(y)dm(y,x)\\ &= \int_B dm'(y,x)\,. \qquad \square\end{aligned}$$

Proof of Theorem 2. Clearly, condition (2) in Theorem 2 implies condition (1). In order to prove the converse let us consider the Markov operator Q defined in the previous lemma and the measure $f^2\mu$ on X. Here we add to the notation for $|A|$ and $|\partial A|$ an index $(Q,f^2\mu)$ in order to distinguish when these notions are used for (P,μ) or for $(Q,f^2\mu)$. One has

$$\begin{aligned} |A|^{(Q,f^2\mu)} &= \int_{\{x\in A\}} f^2(x)d\mu(x) = |A|_{f^2} \\ |\partial A|^{(Q,f^2\mu)} &= \int_{\{x\in A\}}\int_{\{y\in A^c\}} \frac{1}{\lambda} f^{-1}(x)P(x,dy)f(y)f^2(x)d\mu(x) \\ &= \frac{1}{\lambda}\int_{\{x\in A\}}\int_{\{y\in A^c\}} f(x)f(y)P(x,dy)d\mu(x) = \frac{1}{\lambda}|\partial A|_{f^2}\,. \end{aligned}$$

The first condition implies that there exists $c' > 0$ such that

$$c'|\partial A|^{(Q,f^2\mu)} \geq |A|^{(Q,f^2\mu)},$$

which by Proposition 1 implies that

$$\|Q\|_{L^2(X,f^2\mu)\to L^2(X,f^2\mu)} < 1\,.$$

Let $\rho = \|Q\|_{L^2(X,f^2\mu)\to L^2(X,f^2\mu)}$. For any $g \in L^2(X,\mu)$:

$$\begin{aligned} \langle Pg, g\rangle_{L^2(X,\mu)} &= \lambda\left\langle Q\left(\frac{g}{f}\right), \frac{g}{f}\right\rangle_{L^2(X,f^2\mu)} \leq \lambda\rho\left\langle \frac{g}{f}, \frac{g}{f}\right\rangle_{L^2(X,f^2\mu)} \\ &= \lambda\rho\,\langle g, g\rangle_{L^2(X,\mu)}\,. \end{aligned}$$

As P is a self-adjoint operator and $\rho < 1$, this implies

$$\|P\|_{L^2(X,\mu)\to L^2(X,\mu)} < \lambda\,. \qquad \square$$

Proof of Theorem 3. One knows (see Section 3) that P has positive eigenfunctions only for the eigenvalues greater than or equal to $\|P\|$. So the second condition implies the first one.

In order to prove the converse, we remark that for $\gamma \sim \gamma' \in \Gamma$ one has:

$$\begin{aligned} \frac{1}{\lambda(\#S)} f(\gamma') \leq f(\gamma) \leq \lambda(\#S) f(\gamma')\,, \\ P(\gamma,\gamma') = \frac{1}{\#S}\,. \end{aligned}$$

This implies that

$$\frac{1}{\#S}|A|_{f^2} = \sum_{\gamma\in A} f^2(\gamma)\,,$$

$$\frac{1}{\lambda(\#S)^2}|\partial A|_{f^2} \leq \sum_{\gamma\in\partial A} f^2(\gamma) \leq \lambda|\partial A|_{f^2}\,.$$

By Theorem 2, the first condition implies the second one. □

REMARK. The proof of Theorem 3 can easily be generalized to the case where P is a convolution operator with a finitely supported probability measure.

3. REMARKS

We will now make some comments about Theorems 2 and 3. We will state some theorems about the existence of eigenfunctions for the Markov operator and discuss whether one can take in the generalized Følner condition the eigenfunctions to be in $L^2(X,\mu)$.

For simplicity we will suppose that X is a connected, locally finite graph (i.e. the degree of each vertex is finite) and we consider the *simple random walk* going with equal probability from one vertex to any of its neighbors. We associate with this random walk the simple random walk operator P defined by

$$Pf(v) = \frac{1}{N(v)}\sum_{w\sim v} f(w) \quad \text{for } f\in l^2(X,N)$$

where $N(v)$ is the degree of vertex v in X (*i.e.* the number of edges adjacent to v), where $w\sim v$ means that w and v are connected by an edge and where $l^2(X,N)$ is the space of real-valued functions f on the vertices of X such that $\sum_{x\in X} f^2(x)N(x)$ is finite.

3.1 EXISTENCE OF EIGENFUNCTIONS

THEOREM 4 ([20]). *Let X be an infinite, locally finite graph and let P be the simple random walk operator on $l^2(X,N)$. For any $\lambda \geq \|P\|$ there exists a positive eigenfunction f of P with eigenvalue λ, i.e.*

$$Pf(x) = \lambda f(x) \quad \textit{and } f(x) > 0 \textit{ for } x\in X\,.$$

For $\lambda < \|P\|$ there are no positive eigenfunctions of P with eigenvalue λ.