

Appendix: Proof of Proposition 3.1

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APPENDIX: PROOF OF PROPOSITION 3.1

We start with some lemmas.

LEMMA A.1. *Let A be a commutative ring with identity on which a finite group G acts, let A^G be the invariant subring, and let B be a flat A^G -algebra. Then G acts on $B \otimes_{A^G} A$ through its action on the second factor and the invariant subring of this action is B .*

Proof. We have an exact sequence

$$0 \rightarrow A^G \rightarrow A \rightarrow \prod_{g \in G} A,$$

where $\prod_{g \in G} A$ is the direct product of $|G|$ copies of A , and $A \rightarrow \prod_{g \in G} A$ is defined by $a \mapsto (ga - a)$. Since B is a flat A^G -algebra, the tensor product of B with the above sequence remains exact, that is, the sequence

$$0 \rightarrow B \rightarrow B \otimes_{A^G} A \rightarrow \prod_{g \in G} B \otimes_{A^G} A$$

is exact. Hence $B = (B \otimes_{A^G} A)^G$.

Let A be a finitely generated k -algebra on which a finite group G acts. Then A is finite over A^G . For every prime ideal $\mathfrak{q}$ of A^G , let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all the prime ideals of A lying over $\mathfrak{q}$. It is known that G acts transitively on $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Fix a $\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Let $G_d = \{g \in G \mid g\mathfrak{p} = \mathfrak{p}\}$ be the decomposition group at $\mathfrak{p}$.

LEMMA A.2. *Notation as above. Let $\widehat{A^G_{\mathfrak{q}}}$ be the completion of the local ring $A^G_{\mathfrak{q}}$ and let $\widehat{A_{\mathfrak{p}}}$ be the completion of the local ring $A_{\mathfrak{p}}$. Then G_d acts on $\widehat{A_{\mathfrak{p}}}$ and $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G_{\mathfrak{q}}}$.*

Proof. Since $A^G_{\mathfrak{q}}$ is a flat A^G -algebra, we have $(A^G_{\mathfrak{q}} \otimes_{A^G} A)^G = A^G_{\mathfrak{q}}$ by Lemma A.1. Replacing A by $A^G_{\mathfrak{q}} \otimes_{A^G} A$ if necessary, we may thus assume that A^G is a local ring and $\mathfrak{q}$ is the maximal ideal of A^G .

Let $\widehat{A}$ be the completion of A with respect to the $\mathfrak{q}A$ -adic topology. Since A is a finite A^G -algebra, we have $\widehat{A} = \widehat{A^G} \otimes_{A^G} A$. On the other hand, we have $\widehat{A} = \prod_i \widehat{A_{\mathfrak{p}_i}}$. Since $\widehat{A^G}$ is a flat A^G -algebra, we have $\widehat{A^G} = (\widehat{A^G} \otimes_{A^G} A)^G$ by Lemma A.1. So we have $\widehat{A^G} = (\prod_i \widehat{A_{\mathfrak{p}_i}})^G$. Obviously $(\prod_i \widehat{A_{\mathfrak{p}_i}})^G = (\widehat{A_{\mathfrak{p}}})^{G_d}$. Therefore $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G_{\mathfrak{q}}}$.

LEMMA A.3. *Let A be a noetherian local ring, let I_i ($i = 1, \dots, n$) be some ideals of A , and let K_i be the kernel of the canonical homomorphism $\widehat{A} \rightarrow \widehat{A/I_i}$. If $I = I_1 \cdots I_n$, then the kernel of $\widehat{A} \rightarrow \widehat{A/I}$ is $K_1 \cdots K_n$.*

Proof. Since A is noetherian, we have $\ker(\widehat{A} \rightarrow \widehat{A/I_i}) = \widehat{I_i} = I_i \widehat{A}$, that is $K_i = I_i \widehat{A}$. Similarly we have $\ker(\widehat{A} \rightarrow \widehat{A/I}) = I \widehat{A} = I_1 \cdots I_n \widehat{A}$. So $\ker(\widehat{A} \rightarrow \widehat{A/I}) = K_1 \cdots K_n$.

Let T be a k -scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \longrightarrow & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k) . \end{array}$$

We have the following

LEMMA A.4. *Let $s: T \rightarrow X_m \times T$ be a section of q . Then s is a closed immersion and the closed subscheme D defined by s is a relative effective Cartier divisor on $X_m \times T/T$.*

Proof. Since $qs = \text{id}$ is a closed immersion and since q is separated, s is also a closed immersion. The closed subscheme D defined by s is flat because $qs = \text{id}$. Let $\mathcal{I}$ be the sheaf of $\mathcal{O}_{X_m \times T}$ -ideals defining D . We have an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{X_m \times T} \rightarrow \mathcal{O}_D \rightarrow 0 .$$

For any $t \in T$, since $\mathcal{O}_D$ is $\mathcal{O}_T$ flat, the following sequence is exact:

$$0 \rightarrow \mathcal{I} \otimes_{\mathcal{O}_T} k(t) \rightarrow \mathcal{O}_{X_m \times T} \otimes_{\mathcal{O}_T} k(t) \rightarrow \mathcal{O}_{D_t} \rightarrow 0 ,$$

where D_t is the fiber of $D \rightarrow T$ at t . Hence $\mathcal{I} \otimes_{\mathcal{O}_T} k(t)$ is the ideal defining the closed subscheme D_t of $q^{-1}(t) \cong X_m$. Obviously D_t defines a divisor of X_m . So for every point $x \in q^{-1}(t)$, the ideal $\mathcal{I}_x \otimes_{\mathcal{O}_T} k(t)$ of $\mathcal{O}_{X_m \times T, x} \otimes_{\mathcal{O}_T, t} k(t)$ is generated by an element which is not a zero divisor. By Nakayama's lemma, the ideal $\mathcal{I}_x$ of $\mathcal{O}_{X_m \times T, x}$ is generated by one element whose image in $\mathcal{O}_{X_m \times T, x} \otimes_{\mathcal{O}_T, t} k(t)$ is not a zero divisor. By Lemma 2.3, D is a relative effective Cartier divisor.

Consider the sections

$$s_i: (X - S)^n \rightarrow X_m \times (X - S)^n, \quad (P_1, \dots, P_n) \mapsto (P_i, P_1, \dots, P_n).$$

Denote the relative effective Cartier divisors defined by s_i also by s_i , and let $D = s_1 + \dots + s_n$. The relative effective Cartier divisor D can also be regarded as a closed subscheme of $X_m \times (X - S)^n$. The n -th symmetric group $\mathfrak{S}_n$ acts on $(X - S)^n$ by permuting the factors. It acts on $X_m \times (X - S)^n$ through its action on the second factor. Obviously D is stable under this action. Let $\mathcal{D}$ be the quotient of D by $\mathfrak{S}_n$.

PROPOSITION A.5.

- (a) *The quotient of $X_m \times (X - S)^n$ by $\mathfrak{S}_n$ is $X_m \times (X - S)^{(n)}$.*
- (b) *The closed immersion $D \rightarrow X_m \times (X - S)^n$ induces a closed immersion $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$ and $\mathcal{D}$ is a relative effective Cartier divisor on $(X_m \times (X - S)^{(n)})/(X - S)^{(n)}$. Moreover D is the pull-back of $\mathcal{D}$.*

Proof. (a) We have a Cartesian square

$$\begin{array}{ccc} X_m \times (X - S)^n & \longrightarrow & X_m \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ (X - S)^n & \longrightarrow & (X - S)^{(n)}. \end{array}$$

The morphism $X_m \times (X - S)^{(n)} \rightarrow (X - S)^{(n)}$ is flat since it is obtained from the flat morphism $X_m \rightarrow \text{spec}(k)$ through the base extension $(X - S)^{(n)} \rightarrow \text{spec}(k)$. Our assertion then follows directly from Lemma A.1.

(b) Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_m \times (X - S)^n & \longrightarrow & X_m \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ (X - S)^n & \longrightarrow & (X - S)^{(n)}. \end{array}$$

One can easily show that $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$ is a finite morphism and induces a homeomorphism of $\mathcal{D}$ with a closed subset of $X_m \times (X - S)^{(n)}$. We are going to show that for any point $y \in \mathcal{D}$, the homomorphism $\mathcal{O}_{X_m \times (X - S)^{(n)}, y} \rightarrow \mathcal{O}_{\mathcal{D}, y}$ is surjective and the homomorphism $\mathcal{O}_{(X - S)^{(n)}, t} \rightarrow \mathcal{O}_{\mathcal{D}, y}$ is flat, where t is the image of y in $(X - S)^{(n)}$. If this is done, then $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$

is a closed immersion and $\mathcal{D} \rightarrow (X - S)^{(n)}$ is flat. Obviously the fibers of $\mathcal{D} \rightarrow (X - S)^{(n)}$ are effective divisors. As in the proof of Lemma A.4, one can then use Nakayama's lemma and Lemma 2.3 to show that $\mathcal{D}$ is a relative effective Cartier divisor.

One can show that

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong \mathcal{O}_{\mathcal{D},y} \otimes_{\mathcal{O}_{X_m \times (X-S)^{(n)},y}} \widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y}.$$

Note that $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y}$ is a faithfully flat $\mathcal{O}_{X_m \times (X-S)^{(n)},y}$ -algebra. Thus to show that $\mathcal{O}_{X_m \times (X-S)^{(n)},y} \rightarrow \mathcal{O}_{\mathcal{D},y}$ is surjective, it is enough to show that $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y} \rightarrow \widehat{\mathcal{O}}_{\mathcal{D},y}$ is surjective; and to show that $\mathcal{O}_{(X-S)^{(n)},t} \rightarrow \mathcal{O}_{\mathcal{D},y}$ is flat, it is enough to show that $\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \rightarrow \widehat{\mathcal{O}}_{\mathcal{D},y}$ is flat.

Assume $t = n_1 P_1 + \cdots + n_l P_l \in (X - S)^{(n)}$, where the P_i are distinct points of $X - S$, $n_i > 0$ and $\sum_i n_i = n$. Then $y = (P_{i_0}, t) \in X_m \times (X - S)^{(n)}$ for some $i_0 \in \{1, \dots, l\}$. Let $t' = (P_1, \dots, P_1, \dots, P_l, \dots, P_l) \in (X - S)^n$, where the first n_1 components of t' are $P_1, \dots$, and the last n_l components are P_l . The point t' is a point in $(X - S)^n$ lying over $t \in (X - S)^{(n)}$. Let y' be the point (P_{i_0}, t') in $X_m \times (X - S)^n$. It lies over y . Note that y' is also a point in D . With respect to the actions of $\mathfrak{S}_n$ on $(X - S)^n$, on $X_m \times (X - S)^n$, and on D , the decomposition groups at $t' \in (X - S)^n$, at $y' \in X_m \times (X - S)^n$, and at $y' \in D$ are all $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$. We have

$$\widehat{\mathcal{O}}_{(X-S)^n,t} \cong k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{(X-S)^n,t}$ by permuting $x_{i1}, \dots, x_{in_i}$ for each i . We have

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

and the decomposition group $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'}$ by fixing x and permuting $x_{i1}, \dots, x_{in_i}$ for each i .

For each $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \dots, n_1 + \cdots + n_{i_0}\}$, the section

$$s_i: (X - S)^n \rightarrow X_m \times (X - S)^n, \quad (P_1, \dots, P_n) \mapsto (P_i, P_1, \dots, P_n)$$

induces a homomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \rightarrow \widehat{\mathcal{O}}_{(X-S)^n, t'}.$$

Through the isomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]]$$

and the isomorphism

$$\widehat{\mathcal{O}}_{(X-S)^n, t'} \cong k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

this homomorphism induced by s_i is

$$\begin{aligned} k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]] &\rightarrow k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]], \\ x &\mapsto x_{i_0j}, \quad x_{\alpha\beta} \mapsto x_{\alpha\beta} \quad (\alpha = 1, \dots, l, \beta = 1, \dots, n_\alpha), \end{aligned}$$

where $j \in \{1, \dots, n_{i_0}\}$ is uniquely determined by $n_1 + \dots + n_{i_0-1} + j = i$. The kernel of this homomorphism is the ideal $(x - x_{i_0j})$. By Lemma A.3, the kernel of the homomorphism $\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \rightarrow \widehat{\mathcal{O}}_{D, y'}$ is identified with the ideal $\left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right)$ through the isomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]].$$

Hence

$$\widehat{\mathcal{O}}_{D, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right),$$

and the decomposition group $\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_l}$ acts on $\widehat{\mathcal{O}}_{D, y'}$ by fixing x and permuting $x_{i1}, \dots, x_{in_i}$ for each i . Let $\sigma_{i1}, \dots, \sigma_{in_i}$ be the elementary symmetric functions in $x_{i1}, \dots, x_{in_i}$. By Lemma A.2, we have

$$\widehat{\mathcal{O}}_{D, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right),$$

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)}, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]],$$

$$\widehat{\mathcal{O}}_{(X-S)^{(n)}, t} \cong k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

Now it is easy to see that $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)}, y} \rightarrow \widehat{\mathcal{O}}_{D, y}$ is surjective and $\widehat{\mathcal{O}}_{(X-S)^{(n)}, t} \rightarrow \widehat{\mathcal{O}}_{D, y}$ is flat. This proves $\mathcal{D}$ is a relative effective Cartier divisor. We also have

$$\widehat{\mathcal{O}}_{D, y'} = \widehat{\mathcal{O}}_{D, y} \widehat{\otimes}_{\widehat{\mathcal{O}}_{(X-S)^{(n)}, t}} \widehat{\mathcal{O}}_{(X-S)^n, t'}.$$

This implies that $D = \mathcal{D} \times_{(X-S)^{(n)}} (X-S)^n$, that is, D is the pull-back of $\mathcal{D}$. This completes the proof of the proposition.

The relative effective Cartier divisor $\mathcal{D}$ is the universal relative effective Cartier divisor.

LEMMA A.6. *Let T be a k -scheme and let $s_i: T \rightarrow X_m \times T$ ($i = 1, \dots, n$) be some sections of the projection $q: X_m \times T \rightarrow T$. Assume the images of s_i lie in $(X_m - Q) \times T$. Then there is a unique morphism of schemes $f: T \rightarrow (X - S)^{(n)}$ such that the pull-back by $\text{id} \times f$ of the universal relative effective Cartier divisor $\mathcal{D}$ to $X_m \times T$ is $s_1 + \dots + s_n$.*

Proof. Let $p: X_{\mathfrak{m}} \times T \rightarrow X_{\mathfrak{m}}$ be the projection. The morphisms $ps_i: T \rightarrow X_{\mathfrak{m}}$ induce $(ps_1, \dots, ps_n): T \rightarrow X_{\mathfrak{m}}^n$. Since the images of s_i lie in $(X_{\mathfrak{m}} - Q) \times T$, we actually get a morphism $(ps_1, \dots, ps_n): T \rightarrow (X - S)^n$. Composing with the canonical morphism $(X - S)^n \rightarrow (X - S)^{(n)}$, we get $f: T \rightarrow (X - S)^{(n)}$ so that the pull-back of $\mathcal{D}$ by $\text{id} \times f$ is $s_1 + \dots + s_n$. This proves the existence of f .

To prove the uniqueness of f , we first note that $f: T \rightarrow (X - S)^{(n)}$ is uniquely determined as a map on the underlying topological space. Indeed, for every point $t \in T$, $f(t)$ is necessarily the point in $(X - S)^{(n)}$ corresponding to the effective divisor $(s_1 + \cdots + s_n)_t$ on $q^{-1}(t) = X_m$. To prove f is unique as a morphism of schemes, it is enough to prove that the homomorphism on local rings $\mathcal{O}_{(X-S)^{(n)}, f(t)} \rightarrow \mathcal{O}_{T,t}$ induced by f is uniquely determined. It suffices to prove that $\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \rightarrow \widehat{\mathcal{O}}_{T,t}$ is uniquely determined.

Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & (X - S)^{(n)}, \end{array}$$

where D is the closed subscheme of $X_m \times T$ corresponding to the divisor $s_1 + \cdots + s_n$. Let $A = \widehat{\mathcal{O}}_{T,t}$, let $z \in D$ be a point lying over $t \in T$, and let $y \in \mathcal{D}$ be the image of z . We have $\widehat{\mathcal{O}}_{X_m \times T, z} \cong A[[x]]$.

Without loss of generality, assume

$$\begin{aligned} ps_1(t) &= \cdots = ps_{n_1}(t) = P_1, \\ ps_{n_1+1}(t) &= \cdots = ps_{n_1+n_2}(t) = P_2, \end{aligned}$$

$$p_{S_{n_1+\dots+n_{l-1}+1}}(t) = \dots = p_{S_{n_1+\dots+n_l}}(t) = P_l,$$

where $n_i > 0$ ($i = 1, \dots, l$), $n_1 + \dots + n_l = n$, and the P_i are distinct points in $X - S$. Then we have $z = (P_{i_0}, t) \in X_{\mathfrak{m}} \times T$ for some $i_0 \in \{1, \dots, l\}$.

For each $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \dots, n_1 + \cdots + n_{i_0}\}$, the section s_i induces a homomorphism $\widehat{\mathcal{O}}_{X_m \times T, z} \rightarrow \widehat{\mathcal{O}}_{T, t}$, i.e., $A[[x]] \rightarrow A$. Denote the image of x under this homomorphism by $a_{i_0 j}$, where $j \in \{1, \dots, n_{i_0}\}$ is uniquely determined by $n_1 + \cdots + n_{i_0-1} + j = i$. Then by Lemma A.3, we have

$$\widehat{\mathcal{O}}_{D, z} \cong A[[x]] / \left(\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right).$$

Keep the notations in the proof of Proposition A.5. We have

$$\widehat{\mathcal{O}}_{D, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right).$$

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^{(m)}, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

$$\widehat{\mathcal{O}}_{(X-S)^{(m)}, f(t)} \cong k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

We have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{D, z} & \longleftarrow & \widehat{\mathcal{O}}_{D, y} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{O}}_{X_m \times T, z} & \longleftarrow & \widehat{\mathcal{O}}_{X_m \times (X-S)^{(m)}, y} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{O}}_{T, t} & \longleftarrow & \widehat{\mathcal{O}}_{(X-S)^{(m)}, f(t)} \end{array}$$

It is isomorphic to

$$\begin{array}{ccc} A[[x]] / \left(\prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right) & \longleftarrow & k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right) \\ \uparrow & & \uparrow \\ A[[x]] & \longleftarrow & k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \\ \uparrow & & \uparrow \\ A & \longleftarrow & k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]]. \end{array}$$

In order for this last diagram to commute, it is necessary that $\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})$ be mapped to $\prod_{j=1}^{n_{i_0}} (x - a_{i_0j})$ under the homomorphism

$$k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{n_l}]] \rightarrow A[[x]].$$

So the image of σ_{i_0j} under the homomorphism

$$k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{n_l}]] \rightarrow A$$

is necessarily the value at $(a_{i_01}, \dots, a_{i_0n_{i_0}})$ of σ_{i_0j} considered as a function on $A^{n_{i_0}}$. We see that this is true for any indices i_0 and j if we let z go over the points in D above t . Therefore the homomorphism $k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{n_l}]] \rightarrow A$ is uniquely determined, that is, the homomorphism $\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \rightarrow \widehat{\mathcal{O}}_{T,t}$ is uniquely determined. This concludes the proof of the lemma.

LEMMA A.7. *Let T be a k -scheme and let D be a relative effective Cartier divisor on $(X_m \times T)/T$ supported on $(X_m - Q) \times T$ with degree n . Then there exist a flat morphism $T' \rightarrow T$ and sections $s_i: T' \rightarrow X_m \times T'$ ($i = 1, \dots, n$) of the projection $X_m \times T' \rightarrow T'$ such that the pull-back of D to $X_m \times T'$ is equal to $s_1 + \dots + s_n$.*

Proof. By the definition of relative effective Cartier divisors, D is flat over T . On the other hand, $D \rightarrow T$ is proper and has finite fibers. So D is finite over T by [EGA] III, §4.4.2. Take $T_1 = D$. Then we have a finite flat morphism $T_1 \rightarrow T$. Consider the commutative diagram

$$\begin{array}{ccccc} D \times_T T_1 & \xrightarrow{p'} & D & & \\ i' \downarrow & & i \downarrow & & \\ X_m \times T_1 & \xrightarrow{p} & X_m \times T & \longrightarrow & X_m \\ q' \downarrow & & q \downarrow & & \downarrow \\ D = T_1 & \xrightarrow{qi} & T & \longrightarrow & \text{spec}(k). \end{array}$$

Let $\Delta: D \rightarrow D \times_T D = D \times_T T_1$ be the diagonal map. It is a closed immersion since the morphism qi is separated. Take $s_1 = i'\Delta$. This is a section of q' . Hence it defines a relative effective Cartier divisor on $(X_m \times T_1)/T_1$. The pull-back D_1 of the relative effective Cartier divisor D to $X_m \times T_1$ is the closed subscheme defined by i' . Let $\mathcal{I}_{D_1}$ and $\mathcal{I}_s$ be the ideal sheaves of the

closed subschemes defined by i' and s_1 , respectively. Since s_1 factors through i' , we have $\mathcal{I}_{D_1} \subset \mathcal{I}_s$. Hence $D_1 - s$ is a relative effective Cartier divisor on $(X_m \times T_1)/T_1$ by Lemma 2.2 (b), that is, there exists a relative effective Cartier divisor D_1' such that $D_1 = s_1 + D_1'$. Now we take $T_2 = D_1'$. We then have a finite flat morphism $T_2 \rightarrow T_1$, a section $s_2: T_2 \rightarrow X_m \times T_2$ of the projection $X_m \times T_2 \rightarrow T_2$, and a relative effective Cartier divisor D_2' on $(X_m \times T_2)/T_2$ such that the pull-back of D_1' to $X_m \times T_2$ is equal to $s_2 + D_2'$. Then we take $T_3 = D_2'$, In this way we get finite flat morphisms $T_i \rightarrow T_{i-1}$ ($i = 1, \dots, n$), sections $s_i: T_i \rightarrow X_m \times T_i$, such that the pull-back of D to $X_m \times T_n$ is equal to $s_1 + \dots + s_n$, where the s_i denote the relative effective Cartier divisors on $(X_m \times T_n)/T_n$ induced by the sections s_i . This proves our lemma.

Finally we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By Lemma A.7, there exist a finite flat morphism $\pi: T' \rightarrow T$ and sections $s_i: T' \rightarrow X_m \times T'$ ($i = 1, \dots, n$) of the projection $X_m \times T' \rightarrow T'$ such that the pull-back π^*D of D to $X_m \times T'$ is equal to $s_1 + \dots + s_n$. By Lemma A.6, there exists a unique morphism of schemes $f': T' \rightarrow (X - S)^{(n)}$ such that the pull-back $f'^*\mathcal{D}$ of the universal relative effective Cartier divisor $\mathcal{D}$ to $X_m \times T'$ is $s_1 + \dots + s_n$. Let $p_1, p_2: T' \times_T T' \rightarrow T'$ be the projections. We have

$$(f'p_1)^*(\mathcal{D}) = p_1^*f'^*\mathcal{D} = p_1^*(s_1 + \dots + s_n) = p_1^*\pi^*D = p_2^*\pi^*D = \dots = (f'p_2)^*(\mathcal{D}).$$

that is, $(f'p_1)^*(\mathcal{D}) = (f'p_2)^*(\mathcal{D})$. By Lemma A.6 we have $f'p_1 = f'p_2$. By the theory of descent, ([SGA 1] VIII, Theorem 5.2), there exists a unique morphism of schemes $f: T \rightarrow (X_m - Q)^{(n)}$ such that $f' = f\pi$, and the pull-back of $\mathcal{D}$ to $X_m \times T$ is D .

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