

4. The differential equation for the τ -spherical functions

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is an algebra homomorphism of $\mathcal{D}(G; \chi_l)$ into $\mathbf{C}$.

Set

$$(3.21) \quad \alpha_{l,s}(ka_t n) = \frac{1}{d_l} \chi_l(k) e^{-(s+\rho)t}.$$

Since $f = f * d_l \chi_l$ and $\chi_l(k^{-1}) = \chi_l(k)$ for $k \in K$, for every $f \in \mathcal{D}(G; \chi_l)$

$$\begin{aligned} \lambda_s(f) &= \frac{1}{d_l} \int_K \int_{-\infty}^{\infty} \int_N f(ka_t n) \chi_l(k) e^{(-s+\rho)t} dk dt dn \\ &= \int_G f(g) \alpha_{l,s}(g) dg \\ &= \int_G f(g) \int_K \alpha_{l,s}(kgk^{-1}) dk dg \\ (3.22) \quad &= \int_G f(g) \zeta_{l,s}(g) dg \end{aligned}$$

with

$$(3.23) \quad \zeta_{l,s} := \int_K \alpha_{l,s}(kgk^{-1}) dk.$$

One easily checks that $\zeta_{l,s}$ satisfies $\zeta_{l,s} = \zeta_{l,s}^0$, $\zeta_{l,s} * d_l \chi_l = \zeta_{l,s}$ and $\zeta_{l,s}(e) = 1$. Thus $\zeta_{l,s}$ is a τ_l -spherical function. It will be shown in the next section that any τ_l -spherical function is of the form (3.24).

By Remark 2.3, we have

$$(3.24) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \chi_l(k_1) \zeta_{l,s}(a_t) \quad \text{for } g = k_1 k_2 a_t k'_2,$$

so $\zeta_{l,s}$ is uniquely determined by its restriction to A .

4. THE DIFFERENTIAL EQUATION FOR THE τ_l -SPHERICAL FUNCTIONS

For a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$, let $\mathfrak{u}_{\mathbf{C}}$ denote the complex subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by $\mathfrak{u}$. The universal enveloping algebra $\mathcal{U}(\mathfrak{u})$ of $\mathfrak{u}_{\mathbf{C}}$ is considered as a subalgebra of $\mathcal{U}(\mathfrak{g})$.

The representation τ_l of K_1 induces differentiated representations of the Lie algebra $\mathfrak{k}_1$ of K_1 and of the universal enveloping algebra $\mathcal{U}(\mathfrak{k}_1)$ of $(\mathfrak{k}_1)_{\mathbf{C}}$. We indicate these representations with the same letter τ_l . Let $\mathfrak{k}_2$ be the Lie algebra of K_2 . Every element $Y \in \mathfrak{k}_{\mathbf{C}}$ can be uniquely decomposed as

$Y = Y^{(1)} + Y^{(2)}$ with $Y^{(j)} \in (\mathfrak{k}_j)_{\mathbb{C}}$, $j = 1, 2$. The symbol χ_l will also be used for the $\mathbb{C}$ -linear map on $\mathfrak{U}(\mathfrak{k})$ defined by

$$\chi_l(Y_1 \cdots Y_m) := \text{tr} \left[\tau_l(Y_1^{(1)}) \cdots \tau_l(Y_m^{(1)}) \right]$$

for $Y_1, \dots, Y_m \in \mathfrak{k}_{\mathbb{C}}$.

The Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ gives $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{k})\mathfrak{U}(\mathfrak{a})\mathfrak{U}(\mathfrak{n}) \cong \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a}) \oplus \mathfrak{U}(\mathfrak{g})\mathfrak{n}_{\mathbb{C}}$. Let $P: \mathfrak{U}(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{k}) \otimes \mathfrak{U}(\mathfrak{a})$ be the corresponding projection. For $s \in \mathbb{C}$, let e_s be the $\mathbb{C}$ -linear map on $\mathfrak{U}(\mathfrak{a})$ defined by

$$e_s(L^m) := (-1)^m (s + \rho)^m \quad \text{for every integer } m \geq 0.$$

Define $p_{l,s}: \mathfrak{U}(\mathfrak{g}) \rightarrow \mathbb{C}$ to be the composition $p_{l,s} := \left(\frac{1}{d_l} \chi_l \otimes e_s \right) \circ P$, where as before $d_l = \dim \tau_l$.

4.1. PROPOSITION. *Let $\zeta_{l,s}$ be the function defined by Formula (3.23). For every $D \in \mathfrak{U}(\mathfrak{g})^K$ and $g \in G$*

$$(4.25) \quad \zeta_{l,s}(g; D) = p_{l,s}(D) \zeta_{l,s}(g).$$

Proof. Because of Theorem 3.1, $\zeta_{l,s}$ is an eigenfunction of every $D \in \mathfrak{U}(\mathfrak{g})^K$. The eigenvalue corresponding to $D \in \mathfrak{U}(\mathfrak{g})^K$ is $\zeta_{l,s}(e; D)$ because $\zeta_{l,s}(e) = 1$. Since D is K -invariant, $\zeta_{l,s}(e; D) = \alpha_{l,s}(e; D)$. Write $D = \sum_i y_i x_i + \sum_j n_j$ with $y_i \in \mathfrak{U}(\mathfrak{k})$, $x_i \in \mathfrak{U}(\mathfrak{a})$ and $n_j \in \mathfrak{U}(\mathfrak{g})\mathfrak{n}_{\mathbb{C}}$. Then $\alpha_{l,s}(e; D) = \sum_i \alpha_{l,s}(e; y_i x_i)$ because $\alpha_{l,s}(gn) = \alpha_{l,s}(g)$ for $g \in G$ and $n \in N$. To compute $\alpha_{l,s}(e; y_i x_i)$, assume without loss of generality that $x_i = L^{m_i}$ and that $y_i = Y_1 \cdots Y_m$ with $Y_j \in \mathfrak{k}$. The definition of $\alpha_{l,s}$ gives

$$\alpha_{l,s}(e; y_i x_i) = \frac{1}{d_l} \chi_l(y_i) (-1)^m (s + \rho)^m = p_{l,s}(y_i x_i).$$

Thus $\zeta_{l,s}(e; D) = p_{l,s}(D)$. $\square$

Let $\delta_l(D)$ denote the τ_l -radial component on $A^+ := \{a_t : t > 0\}$ of the differential operator $D \in \mathfrak{U}(\mathfrak{g})$; that is, the unique differential operator on A^+ satisfying

$$f(a_t; \delta_l(D)) = f(a_t; D)$$

for all $f \in \mathcal{D}(G; \chi_l)$ and $t > 0$. Proposition 4.1 immediately implies

4.2. COROLLARY. *$\zeta_{l,s}$ is an eigenfunction of the τ_l -radial component on A^+ of every K -invariant differential operator on G :*

$$(4.26) \quad \zeta_{l,s}(a_t; \delta_l(D)) = p_{l,s}(D) \zeta_{l,s}(a_t) \quad (D \in \mathfrak{U}(\mathfrak{g})^K, t > 0).$$

We now write (4.26) explicitly in the case D is the Casimir operator ω of $\mathfrak{g}$. Let B denote the Cartan-Killing form of $\mathfrak{g}_{\mathbf{C}}$ ($\cong \mathfrak{sp}(1+n, \mathbf{C})$). If $X, Y \in \mathfrak{sp}(1, n)$, then

$$B(X, Y) = 4(n+2) \Re \operatorname{tr}(XY)$$

where $\Re$ denotes the quaternionic real part: $\Re q = \frac{q+\bar{q}}{2}$ for $q \in \mathbf{H}$. The bilinear form $B_{\theta}(X, Y) := -B(X, \theta Y)$ is an inner product on $\mathfrak{g}$. Orthonormality will be considered with respect to B_{θ} .

Let $\{Z_j\}_{j=1}^m$ ($m := 2n^2 + n$) and $\{X_{\beta,j}\}_{j=1}^{m_{\beta}}$ ($\beta \in \{\alpha, 2\alpha\}$) be orthonormal bases in $\mathfrak{m}$ and in $\mathfrak{g}_{\beta}$ respectively. Define $X_{-\beta,j} = -\theta(X_{\beta,j})$ for $\beta \in \{\alpha, 2\alpha\}$ and $j = 1, \dots, m_{\beta}$. Then $\{X_{-\beta,j}\}_{j=1}^{m_{\beta}}$ is an orthonormal basis for $\mathfrak{g}_{-\beta}$, and $B(X_{\beta,i}, X_{-\beta,j}) = \delta_{ij}$. Moreover, for all $j = 1, \dots, m_{\beta}$, $H_{\beta} := [X_{\beta,j}, X_{-\beta,j}]$ is the unique element of $\mathfrak{a}$ satisfying $B(H_{\beta}, L) = \beta(L)$, i.e.

$$H_{\beta} = \frac{h_{\beta}}{8(n+2)} L \quad \text{with} \quad h_{\beta} = \begin{cases} 1 & \text{if } \beta = \alpha \\ 2 & \text{if } \beta = 2\alpha. \end{cases}$$

Set $H_1 := \frac{L}{\sqrt{8(n+2)}}$, a unit vector in $\mathfrak{a}$. Then, if $D_{\beta,j} := X_{\beta,j}X_{-\beta,j} + X_{-\beta,j}X_{\beta,j}$, we have (cf. [GaV], p. 132)

$$\begin{aligned} (4.27) \quad \omega &= \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta \in \{\alpha, 2\alpha\}} \sum_{j=1}^{m_{\beta}} D_{\beta,j} \\ &= \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta \in \{\alpha, 2\alpha\}} m_{\beta} H_{\beta} + 2 \sum_{\beta \in \{\alpha, 2\alpha\}} \sum_{j=1}^{m_{\beta}} X_{\beta,j} X_{-\beta,j} \end{aligned}$$

where

$$(4.28) \quad \omega_{\mathfrak{m}} := - \sum_{j=1}^m Z_j^2.$$

Hence

$$P(\omega) = \omega_{\mathfrak{m}} + H_1^2 + \sum_{\beta \in \{\alpha, 2\alpha\}} m_{\beta} H_{\beta} = \omega_{\mathfrak{m}} + \frac{L^2 + 2\rho L}{B(L, L)},$$

from which we conclude

$$(4.29) \quad p_{l,s}(\omega) = p_{l,s}(\omega_{\mathfrak{m}}) + \frac{(s+\rho)^2 - 2\rho(s+\rho)}{B(L, L)} = \frac{1}{d_l} \chi_l(\omega_{\mathfrak{m}}) + \frac{s^2 - \rho^2}{8(n+2)}.$$

To compute $\delta_l(\omega)$ we use Formula (4.27). Observe first that if $f \in \mathcal{D}(G; \chi_l)$ and $Y \in \mathfrak{U}(\mathfrak{k})$, then $f(a_t; Y) = \frac{1}{d_l} \chi_l(Y)$. Hence $\delta_l(Y) = \frac{1}{d_l} \chi_l(Y)$. In particular,

$$(4.30) \quad \delta_l(\omega_m) = \frac{1}{d_l} \chi_l(\omega_m).$$

Write

$$(4.31) \quad X_{\beta,j} = Y_{\beta,j} + P_{\beta,j} \quad \text{with} \quad Y_{\beta,j} \in \mathfrak{k}, P_{\beta,j} \in \mathfrak{p}.$$

A standard computation (cf. e.g. [W2], p.278) then gives for $f \in \mathcal{D}(G; \chi_l)$ and $t > 0$

$$f(a_t; D_{\beta,j}) = \coth(t\beta(L))f(a_t; H_\beta) + \frac{4}{d_l} \frac{1 - \cosh(t\beta(L))}{\sinh^2(t\beta(L))} \chi_l(Y_{\beta,j}^2) f(a_t)$$

i.e.

$$(4.32) \quad \delta_l(D_{\beta,j}) = \coth(t\beta(L))H_\beta + \frac{4}{d_l} \frac{1 - \cosh(t\beta(L))}{\sinh^2(t\beta(L))} \chi_l(Y_{\beta,j}^2).$$

Notice that $\chi_l(Y_{\alpha,j}^2) = 0$ for all $j = 1, \dots, m_\alpha$.

For $h = i, j, k$, let Y_h denote the tangent vector at e to the 1-parameter subgroup $t \mapsto \cos t + h \sin t$ in $\mathrm{Sp}(1)$. Explicit choices of the orthonormal bases in $\mathfrak{m}$ and $\mathfrak{g}_{2\alpha}$ prove that

$$(4.33) \quad \chi_l(\omega_m) = -2 \sum_{j=1}^3 \chi_l(Y_{2\alpha,j}^2) = -\frac{1}{8(n+2)} \sum_{h \in \{i,j,k\}} \mathrm{tr} [\tau_l(Y_h)^2].$$

As shown in [T1], p.381, there exists an orthonormal basis $\{v_p\}_{p=-l}^l$ in V_l such that

$$\begin{aligned} \tau_l(Y_i)v_p &= -2ipv_p \\ \tau_l(Y_j)v_p &= -i\alpha_{p+1}^l v_{p+1} \\ \tau_l(Y_k)v_p &= -\alpha_{p+1}^l v_{p+1} + \alpha_p^l v_{p-1} \end{aligned}$$

where

$$\alpha_p^l := [(l+p)(l-p+1)]^{1/2}.$$

It follows that for $h = i, j, k$

$$(4.34) \quad \mathrm{tr} [\tau_l(Y_h)^2] = -\frac{4}{3} l(l+1)(2l+1).$$

Identify A with $\mathbf{R}$ and L with $\frac{d}{dt}$ under the isomorphism $t \mapsto \exp(tL) = a_t$. Formulas (4.27), (4.30) and (4.32)–(4.34) then prove the following proposition.

4.3. PROPOSITION. *Let τ_l be an irreducible unitary representation of K_1 of dimension $2l + 1$. Then*

1. *The τ_l -radial component of the Casimir operator ω is*

$$\delta_l(\omega) = \frac{1}{8(n+2)} \left\{ \frac{d^2}{dt^2} + [(4n-1) \coth t + 3 \tanh t] \frac{d}{dt} + \frac{4l(l+1)}{\cosh^2 t} + 4l(l+1) \right\}.$$

2. *For every $s \in \mathbf{C}$*

$$(4.35) \quad p_{l,s}(\omega) = \frac{1}{8(n+2)} [4l(l+1) + s^2 - \rho^2].$$

3. *For every $s \in \mathbf{C}$, the function $\zeta_{l,s}(t) := \zeta_{l,s}(a_t)$ satisfies the differential equation on $(0, +\infty)$*

$$(4.36) \quad u'' + [(4n-1) \coth t + 3 \tanh t] u' + \frac{4l(l+1)}{\cosh^2 t} u = (s^2 - \rho^2) u.$$

The substitution $v(t) = (\cosh t)^{-2l} u(t)$ transforms the differential equation (4.36) into the Jacobi differential equation (cf. [K2], p. 6)

$$(4.37) \quad v'' + [(4n-1) \coth t + (4l+3) \tanh t] v' = (s^2 - \tilde{\rho}^2) v$$

with parameters $\alpha = 2n - 1$ and $\beta = 2l + 1$. Here $\tilde{\rho} := \alpha + \beta + 1 = \rho + 2l$. The Jacobi function

$$(4.38) \quad \begin{aligned} \phi_{is}^{(2n-1, 2l+1)}(t) &:= F\left(\frac{\tilde{\rho} + s}{2}, \frac{\tilde{\rho} - s}{2}; 2n; -\sinh^2 t\right) \\ &= F\left(\frac{\rho + s}{2} + l, \frac{\rho - s}{2} + l; 2n; -\sinh^2 t\right) \end{aligned}$$

is the unique solution v to (4.37) satisfying $v(0) = 1$, $v'(0) = 0$. (In (4.38), $F(a, b; c; z)$ denotes the analytic branch on $\mathbf{C} \setminus [1, \infty)$ of the hypergeometric function.)

The τ_l -spherical function $\zeta_{l,s}$ is therefore explicitly given by

$$(4.39) \quad \begin{aligned} \zeta_{l,s}(t) &:= \zeta_{l,s}(a_t) = (\cosh t)^{2l} \phi_{is}^{(2n-1, 2l+1)}(t) \\ &= (\cosh t)^{2l} F\left(\frac{\rho + s}{2} + l, \frac{\rho - s}{2} + l; 2n; -\sinh^2 t\right). \end{aligned}$$

Formula (4.39) has been previously determined by Takahashi ([T2], Formula (7), p. 225) by direct integration of (3.23), using the following expression of χ_l in terms of Gegenbauer polynomials:

$$(4.40) \quad \chi_l(k_1) = C_{2l}^1(\Re k_1) = \frac{\sin((2l+1)\vartheta)}{\sin \vartheta} \quad \text{if } \Re k_1 = \cos \vartheta.$$

Formula (4.35) shows that $p_{l,s}(\omega)$ is an even function of s which assumes arbitrary complex values as s varies in $\mathbf{C}$. The following corollary can therefore be deduced from Theorem 3.1 and Proposition 4.3.

4.4. COROLLARY. *The τ_l -spherical functions are exactly the functions $\{\zeta_{l,s} : s \in \mathbf{C}\}$ given by Formulas (3.24) and (4.39). Further, $\zeta_{l,s}$ satisfies $\zeta_{l,s}(g) = \zeta_{l,s}(g^{-1})$ for all $g \in G$. Moreover, $\zeta_{l,s} = \zeta_{l,s'}$ if and only if $s = \pm s'$.*

The functional equation (3.15) with $g_1 = a_t$ and $g_2 = a_\tau$ becomes (cf. [T2], Théorème 1, p. 227)

$$(4.41) \quad \zeta_{l,s}(t)\zeta_{l,s}(\tau) = \int_0^\infty K_l(t, \tau, u)\zeta_{l,s}(u)\Delta(u) du$$

where Δ is as in (1.7) and the kernel $K_l(t, \tau, u)$ is defined as follows. Set

$$B := \frac{\cosh^2 t + \cosh^2 \tau + \cosh^2 u - 1}{2 \cosh t \cosh \tau \cosh u}.$$

Then

$$(4.42) \quad K_l(t, \tau, u) := \frac{2^{-2\rho}\Gamma(2n)}{\sqrt{\pi}\Gamma(2n - \frac{1}{2})} \frac{(\cosh t \cosh \tau \cosh u)^{2n-3}}{(\sinh t \sinh \tau \sinh u)^{4n-2}} (1 - B^2)^{2n-\frac{3}{2}} \\ \times F\left(2n + 2l, 2n - 2l - 2; 2n - \frac{1}{2}; \frac{1}{2}(1 - B)\right)$$

if $B < 1$, and $K_l(t, \tau, u) := 0$ if $B \geq 1$. Using (4.39) and Formula (7.11) in [K2], one can prove that (4.41) holds also outside our group-theoretical setting for all $l \in \mathbf{R}$ satisfying $2n - 1 > 2l \geq 0$.

5. THE POSITIVE DEFINITE τ_l -SPHERICAL FUNCTIONS

A continuous function ζ on a locally compact group G is said to be *positive definite* if for every $f \in C_c(G)$

$$\int_G \int_G \zeta(x^{-1}y) f(x) \overline{f(y)} dx dy \geq 0.$$

In this section we establish which among the $\zeta_{l,s}$ are positive definite.

Let us first introduce some notation and recall some definitions. Let G be a semisimple Lie group with finite center, and let K be a maximal compact subgroup of G . $\mathfrak{g}$ and $\mathfrak{k}$ ($\subset \mathfrak{g}$) are the Lie algebras of G and K , respectively. A (strongly continuous) representation T of G on a Banach space $\mathcal{H}$ is denoted by $(T, \mathcal{H})$. We may simply speak of the representation T if $\mathcal{H}$ is understood. Irreducibility for T always means topological irreducibility (= no closed proper invariant subspaces). Let $\widehat{K}$ denote the set of equivalence classes