

1. The fine structure of $Sp(1,n)$

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

In Section 6 we prove that the τ -Abel transform is an isomorphism of $\mathcal{D}(G; \chi_\tau)$ onto the convolution algebra $\mathcal{D}_+(\mathbf{R})$ of the even C^∞ compactly supported functions on $\mathbf{R}$. The inversion formula is explicitly written. The Paley-Wiener Theorem for the τ -spherical transform is an immediate consequence. The final Section 7 contains the inversion formula and the Plancherel Theorem for the τ -spherical transform.

Similar results for $SU(n, 1)$ have been obtained as a specialization of the Hermitian symmetric case by Shimeno [Shi] and Heckman [HS, Part 1].

ACKNOWLEDGMENT. During the preparation of this paper, the second author has been financially supported by the Dutch Organization for Scientific Research (N.W.O.).

1. THE FINE STRUCTURE OF $Sp(1, n)$

Let $\mathbf{H}$ be the skew-field of the quaternions. Consider on the right $\mathbf{H}$ -vector space $\mathbf{H}^{n+1}$ the Hermitian form

$$(1.1) \quad [x, y] = \bar{y}_0 x_0 - \bar{y}_1 x_1 - \cdots - \bar{y}_n x_n,$$

the bar sign denoting quaternionic conjugation: if $1, i, j, k$ are the quaternionic units and $q = a + ib + jc + kd \in \mathbf{H}$ (with $a, b, c, d \in \mathbf{R}$), then $\bar{q} = a - ib - jc - kd$. Let $G = Sp(1, n)$ be the group $U(1, n; \mathbf{H})$ of $(n+1) \times (n+1)$ matrices with coefficients in $\mathbf{H}$ which preserve this form. For $n = 1$, G is called the De Sitter group. Let $Sp(m)$ indicate the group $U(m; \mathbf{H})$ of $m \times m$ matrices with coefficients in $\mathbf{H}$ which preserve the inner product $(x, y) = \bar{y}_1 x_1 + \cdots + \bar{y}_m x_m$ of $\mathbf{H}^m$. In particular, $Sp(1)$ consists of the quaternions $q = a + ib + jc + kd$ with norm $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$ equal to 1. $Sp(1)$ is canonically isomorphic to $SU(2)$. The group G acts on the projective space $P_n(\mathbf{H})$. Let Ω denote the image of the open set $\{x \in \mathbf{H}^{n+1} : [x, x] > 0\}$ under the canonical map $\mathbf{H}^{n+1} \setminus \{0\} \rightarrow P_n(\mathbf{H})$. Then G acts transitively on Ω , and the stabilizer of the quaternionic line generated by the vector $(1, 0, \dots, 0)$ is the group

$$K = \left\{ \begin{bmatrix} u & 0 \\ 0 & U \end{bmatrix} : u \in Sp(1), U \in Sp(n) \right\} \equiv Sp(1) \times Sp(n).$$

The homogeneous space G/K is called the hyperbolic quaternionic space. K is a maximally compact subgroup of G . G is connected and simply connected.

To study the fine structure of G , we consider its Lie algebra $\mathfrak{g} = \mathfrak{sp}(1, n)$. Let J be the $(n+1) \times (n+1)$ matrix $\mathrm{diag}(-1, 1, \dots, 1)$. For any matrix X of type $(n+1, n+1)$ with coefficients in $\mathbf{H}$ we set $X^* = J\bar{X}^t J$, the symbol t denoting transposition.

The Lie algebra $\mathfrak{g}$ consists of the matrices X which verify the relation

$$X + X^* = 0.$$

These are the matrices of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ \bar{Z}_2^t & Z_3 \end{bmatrix}$$

with Z_1 and Z_3 anti-Hermitian of type $(1, 1)$ and (n, n) , respectively, and Z_2 arbitrary. Let θ be the anti-involutive automorphism of $\mathfrak{g}$ defined by

$$\theta X = JXJ.$$

Then θ is a Cartan involution with the usual decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Here $\mathfrak{k}$ is the Lie algebra of K . Let L be the following element of $\mathfrak{g}$:

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathbf{0} & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $L \in \mathfrak{p}$ and $\mathfrak{a} = \mathbf{R}L$ is a maximal Abelian subspace of $\mathfrak{p}$. We are going to diagonalize $\mathrm{ad}L$. The centralizer of L in $\mathfrak{k}$ is the subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ of the matrices

$$\begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with $u \in \mathbf{H}$, $u + \bar{u} = 0$ and V a matrix of type $(n-1, n-1)$ satisfying $V + \bar{V}^t = \mathbf{0}$. The non-zero eigenvalues of $\mathrm{ad}L$ are $\alpha = 1$, $-\alpha$, $\pm 2\alpha$. The space $\mathfrak{g}_\alpha$ consists of the matrices

$$X = \begin{bmatrix} 0 & z^* & 0 \\ z & \mathbf{0} & -z \\ 0 & z^* & 0 \end{bmatrix}$$

where z is a matrix of type $(n-1, 1)$ with coefficients in $\mathbf{H}$, and $z^* := \bar{z}^t$. The real dimension of $\mathfrak{g}_\alpha$ is $m_\alpha = 4(n-1)$. The space $\mathfrak{g}_{2\alpha}$ consists of the matrices of the form

$$X = \begin{bmatrix} w & 0 & -w \\ 0 & \mathbf{0} & 0 \\ w & 0 & -w \end{bmatrix}$$

with $w \in \mathbf{H}$, $w + \bar{w} = 0$. The dimension of $\mathfrak{g}_{2\alpha}$ is equal to $m_{2\alpha} = 3$. We have $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$.

Let A be the subgroup $\exp \mathfrak{a}$. This is the subgroup of the matrices

$$a_t = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}$$

where t is a real number. The centralizer of A in K is the subgroup M of the matrices

$$m(u, V) = \begin{bmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{bmatrix}$$

with $u \in \mathrm{Sp}(1)$ and $V \in \mathrm{Sp}(n-1)$. The Lie algebra of M is $\mathfrak{m}$. The subspace $\mathfrak{n} = \mathfrak{g}_{\alpha} + \mathfrak{g}_{2\alpha}$ is a (real) nilpotent subalgebra. Set $N = \exp \mathfrak{n}$. This is the subgroup of the matrices

$$n(w, z) = \begin{bmatrix} 1 + w - \frac{1}{2}[z, z] & z^* & -w + \frac{1}{2}[z, z] \\ z & I & -z \\ w - \frac{1}{2}[z, z] & z^* & 1 - w + \frac{1}{2}[z, z] \end{bmatrix}$$

where $w \in \mathbf{H}$ satisfies $w + \bar{w} = 0$ and $z = [z_1, \dots, z_{n-1}]^t$ is a matrix of type $(n-1, 1)$ with coefficients in $\mathbf{H}$. We have set $z^* = \bar{z}^t$ and $[z, z] = -\bar{z}_1 z_1 - \dots - \bar{z}_{n-1} z_{n-1}$.

The composition law in N is the following:

$$n(w, z) \cdot n(w', z') = n(w + w' + \Im[z, z'], z + z'),$$

where $\Im q := \frac{q - \bar{q}}{2}$ for $q \in \mathbf{H}$. The subgroups A and M normalize N :

$$\begin{aligned} a_t n(w, z) a_{-t} &= n(e^{2t} w, e^t z), \\ m(u, V) n(w, z) m(u, V)^{-1} &= n(uw\bar{u}, Vz\bar{u}). \end{aligned}$$

Let 2ρ be the trace of the restriction of $\mathrm{ad} L$ to $\mathfrak{n}$:

$$(1.2) \quad \rho = \frac{1}{2}(m_{\alpha} + 2m_{2\alpha}) = 2n + 1.$$

We have the Iwasawa decomposition $G = KAN = KNA$ and the corresponding integral formulas:

$$(1.3) \quad \int_G f(g) dg = \int_K \int_{-\infty}^{+\infty} \int_N f(ka_t n) e^{2\rho t} dk dt dn$$

$$(1.4) \quad = \int_K \int_N \int_{-\infty}^{+\infty} f(kna_t) dk dn dt$$

for $f \in C_c(G)$. We adopt the usual notation $C_c(G)$ for the space of continuous functions on G with compact support. In the above formulas, $dn = dw dz$ ($n = n(w, z)$) and dk is the normalized Haar measure on K .

Let

$$K_1 = \left\{ \begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} : u \in \mathrm{Sp}(1) \right\}, \quad K_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} : U \in \mathrm{Sp}(n) \right\}.$$

Then every $g \in G$ can be written as $g = k_1 k_2 a_t k'_2$ for uniquely determined $k_1 \in K_1$, $t \geq 0$ and for some $k_2, k'_2 \in K_2$. Writing $g = [g_{ij}]_{i,j=0}^n$, we have

$$(1.5) \quad k_1 = \frac{g_{00}}{|g_{00}|} \quad \text{and} \quad \cosh t = |g_{00}|.$$

If $g \notin K$, then $t > 0$ and k_2, k'_2 are uniquely determined modulo the subgroup

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & 1 \end{bmatrix} : V \in \mathrm{Sp}(n-1) \right\}.$$

After dg is normalized according to (1.3), the corresponding integral formula is

$$(1.6) \quad \int_G f(g) dg = \frac{1}{2} \left(\frac{\pi}{4} \right)^{2n} \frac{1}{\Gamma(2n)} \int_{K_1} \int_{K_2} \int_0^\infty \int_{K_2} f(k_1 k_2 a_t k'_2) \Delta(t) dk_1 dk_2 dt dk'_2$$

where

$$(1.7) \quad \Delta(t) := 2^{2\rho} (\sinh t)^{4n-1} (\cosh t)^3.$$

2. THE CONVOLUTION ALGEBRA $\mathcal{D}(G; \chi_l)$

Let $\mathbf{N}/2$ be the set of nonnegative half-integers $\{0, 1/2, 1, 3/2, \dots\}$. Since $K_1 \cong \mathrm{Sp}(1)$ is isomorphic to $\mathrm{SU}(2)$, $\mathbf{N}/2$ parametrizes the set of equivalence classes of unitary irreducible representations of K_1 . We denote with the same symbol τ_l either the equivalence class corresponding to the parameter l or a fixed representative for it. Thus τ_l is a unitary irreducible representation of K_1 in a Hilbert space V_l of dimension $d_l = 2l + 1$. We extend τ_l to a representation of K by setting $\tau_l \equiv \mathbf{1}$ on K_2 . Each τ_l is self-dual, i.e. unitarily equivalent to its contragredient representation. It follows in particular that the character $\chi_l = \mathrm{tr} \tau_l$ of τ_l satisfies $\chi_l(k^{-1}) = \chi_l(k)$, $k \in K$.

We denote by $\mathcal{D}(G; \tau_l)$ the convolution algebra of the compactly supported C^∞ maps $F: G \rightarrow \mathrm{End}(V_l)$ satisfying