Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.04.2024**

L'Enseignement Mathématique, t. 45 (1999), p. 219–252

HARMONIC ANALYSIS ON VECTOR BUNDLES OVER $\mathrm{Sp}(1,n)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$

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ABSTRACT. Harmonic analysis on vector bundles over $\mathrm{Sp}(1,n)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ associated with a finite dimensional representation τ of $\mathrm{Sp}(1)$ is developed using Godement's approach of trace spherical functions. The trace spherical trace functions are written in terms of Jacobi functions, and among them the positive definite ones are singled out. An inversion formula for the generalized Abel transform is given explicitly. The Paley-Wiener theorem, the inversion formula and the Plancherel theorem for the τ-spherical transform are determined.

INTRODUCTION

Harmonic analysis over Riemannian symmetric spaces of noncompact type is a fundamental and powerful area of mathematics that exhibits a beautiful interplay between the theory of special functions and the representation theory of semisimple Lie groups. Grown around the monumental work of Harish-Chandra, it has nowadays reached a nearly complete formulation, but, in its development, it has also laid the foundations of its natural extension: harmonic analysis on vector bundles over Riemannian symmetric spaces of noncompact type. Motivated also by many physical applications, this new subject is currently studied very intensively (cf. for instance [BR], [O], [Shi], [Cam], [P], [vdV], [M], [Dei], [BOS]), but a general theory has not yet been formulated.

In this paper we present a complete treatment of harmonic analysis for the spherical transform on a certain class of vector bundles over the hyperbolic space $\mathrm{Sp}(1,n)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$. Set $G = \mathrm{Sp}(1,n)$ and $K = \mathrm{Sp}(1) \times \mathrm{Sp}(n)$. The class of vector bundles we consider are those associated with finite-dimensional irreducible representations τ of K which are trivial on $\mathrm{Sp}(n)$, so actually finite dimensional representations of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. This setting is sufficiently general to exhibit the new features of the theory, namely representations τ of arbitrary dimensions and the possible occurrence of the discrete series in the Plancherel formula. But, at the same time, it is also very concrete and therefore allows us to determine very explicit formulas, which make this paper also a ready-to-use reference for applications of harmonic analysis. In fact, this work is a prelude to the canonical representations of $\mathrm{Sp}(1,n)$ associated with representations of $\mathrm{Sp}(1)$ (cf. [DP]). In a special case, these representations have been studied by van Dijk and Hille [DH], but the introduction of canonical representations goes back to Berezin and to Gel′fand, Graev and Vershik. The main task is their decomposition into irreducible components, and, for this purpose, one needs the harmonic analysis we have developed in this paper.

The methods we employ take their roots in the work of Harish-Chandra and Godement, but their particular application we consider appears to be new. We have tried to keep a down-to-earth exposition in order to make the deep work of these authors accessible to a large mathematical audience. We have adopted Godement's prospective of trace spherical functions, but other points of view are possible (see [Dij]). Partial results have been previously obtained by Takahashi (but neither the Plancherel formula, nor the list of the positive definite spherical functions) and by Ørsted and Zhang (only – incomplete – results on the Plancherel formula). The Plancherel formula has been recently determined by Camporesi [Cam] in a much wider context than ours. His formula is however of very little use for practical purposes, and it does not even transparently show the possible splitting of the spectrum into continuous and discrete parts. Moreover, it has required the full Plancherel theorem on G, a tool by far more complicated than those employed for the known Plancherel theorem for the K-bi-invariant functions.

Let us now describe in more detail the background of the paper.

In [Go], Godement developed a general theory for the functions on a locally compact group G which are spherical with respect to a compact subgroup K. In his definition, the spherical functions on G arise from K-finite irreducible Banach representations of G. Let $g \mapsto T(g)$ be such a representation of G on $\mathcal{H}$. Suppose τ is an equivalence class of irreducible unitary representations of K that occurs in the restriction $T|_K$ of T to K. Let d_τ and χ_τ respectively denote the dimension and the character of τ. Set $\xi_\tau(k) = d_\tau \chi_\tau(k^{-1})$ for $k \in K$, and form the projection $E(\tau) = T|_K(\xi_\tau)$ of $\mathcal{H}$ onto the K-isotypic subspace of $\mathcal{H}$ of type τ. Then the spherical trace function $\zeta_{\tau,T}$ of type τ (shortly, τ-spherical function) on G associated with T is defined as the trace

$$\zeta_{\tau,T}(g) = \frac{1}{d_\tau}\ \mathrm{tr}[E(\tau)T(g)E(\tau)]\,.$$

$\zeta_{\tau,T}$ is said to be of height p when τ occurs p times ($p \geq 1$) in $T|_K$.

Suppose, as in our situation, that G is a semisimple Lie group. Then the τ-spherical functions of height 1 have much more manageable descriptions, either as common eigenfunctions of the left-invariant differential operators on G which are $\mathrm{Ad}(K)$-invariant, or as characters of the convolution algebra $\mathcal{D}(G;\chi_\tau)$ of all C^∞ compactly supported K-central functions on G with fixed K-type τ. Moreover, as proved by Godement, all τ-spherical functions are of height 1 when $\mathcal{D}(G;\chi_\tau)$ is commutative. In this case, the τ-spherical functions are the building blocks for the harmonic analysis on L^2-sections of the homogeneous vector bundle on G/K associated with τ. For example, the algebra $\mathcal{D}(G;\chi_\tau)$ is always commutative when G has finite center, K is maximally compact in G and τ is the trivial representation $\mathbf{1}$. Then the spherical functions of type $\mathbf{1}$ agree with the usual K-bi-invariant spherical functions on G. A less classical example of commutativity of $\mathcal{D}(G;\chi_\tau)$ is provided by Hermitian symmetric pairs (G,K) and 1-dimensional representations of K (cf. [Shi]).

Takahashi recognized in [T2] that if $G=\mathrm{Sp}(1,n)$ and $K=\mathrm{Sp}(1)\times\mathrm{Sp}(n)$, then the algebra $\mathcal{D}(G;\chi_\tau)$ is commutative for every irreducible representation τ of K which is trivial on $\mathrm{Sp}(n)$. He also explicitly computed some characters of $\mathcal{D}(G;\chi_\tau)$ (and it turns out that they are all !) in terms of Jacobi functions. The case $n=1$ has been previously studied by the same author in [T1].

Our paper is organized as follows. In Section 1 we recall some structural properties of $\mathrm{Sp}(1,n)$. Section 2 describes the commutativity of the algebra $\mathcal{D}(G;\chi_\tau)$ associated with a representation τ of K which is trivial on $Sp(n)$. Section 3 introduces the τ-spherical functions as characters of $\mathcal{D}(G;\chi_\tau)$ and gives their first properties. In Section 4 we find the differential equations satisfied by the spherical functions. This either provides us with their explicit expression in terms of Jacobi functions, or allows us to conclude that they are indeed all the spherical functions for $\mathrm{Sp}(1,n)$ associated with the given representation of $\mathrm{Sp}(1)\subset K$.

In Section 5 we write the spherical functions of type τ as the trace of the projection on the K-type τ of certain degenerate principal series representations of $\mathrm{Sp}(1,n)$ which have been studied by Howe and Tan [HT]. From this we can establish which among our spherical functions are positive definite. We underline the occurrence of a rather peculiar phenomenon: for a fixed representation τ, there are positive definite spherical functions arising from the complementary series of $\mathrm{Sp}(1,n)$ if and only if there are no positive definite spherical functions arising from the discrete series.

In Section 6 we prove that the τ-Abel transform is an isomorphism of $\mathcal{D}(G;\chi_\tau)$ onto the convolution algebra $\mathcal{D}_+(\mathbf{R})$ of the even C^∞ compactly supported functions on $\mathbf{R}$. The inversion formula is explicitly written. The Paley-Wiener Theorem for the τ-spherical transform is an immediate consequence. The final Section 7 contains the inversion formula and the Plancherel Theorem for the τ-spherical transform.

Similar results for $\mathrm{SU}(n,1)$ have been obtained as a specialization of the Hermitian symmetric case by Shimeno [Shi] and Heckman [HS, Part 1].

ACKNOWLEDGMENT. During the preparation of this paper, the second author has been financially supported by the Dutch Organization for Scientific Research (N.W.O.).

1. THE FINE STRUCTURE OF $\mathrm{Sp}(1,n)$

Let $\mathbf{H}$ be the skew-field of the quaternions. Consider on the right $\mathbf{H}$-vector space $\mathbf{H}^{n+1}$ the Hermitian form

$$[x,y] = \bar{y}_0 x_0 - \bar{y}_1 x_1 - \cdots - \bar{y}_n x_n \,, \tag{1.1}$$

the bar sign denoting quaternionic conjugation: if $1, i, j, k$ are the quaternionic units and $q = a + ib + jc + kd \in \mathbf{H}$ (with $a,b,c,d \in \mathbf{R}$), then $\bar{q} = a - ib - jc - kd$. Let $G = \mathrm{Sp}(1,n)$ be the group $\mathrm{U}(1,n;\mathbf{H})$ of $(n+1)\times(n+1)$ matrices with coefficients in $\mathbf{H}$ which preserve this form. For $n = 1$, G is called the De Sitter group. Let $\mathrm{Sp}(m)$ indicate the group $\mathrm{U}(m;\mathbf{H})$ of $m \times m$ matrices with coefficients in $\mathbf{H}$ which preserve the inner product $(x,y) = \bar{y}_1 x_1 + \cdots + \bar{y}_m x_m$ of $\mathbf{H}^m$. In particular, $\mathrm{Sp}(1)$ consists of the quaternions $q = a + i\,b + j\,c + k\,d$ with norm $|q| = \sqrt{a^2+b^2+c^2+d^2}$ equal to 1. $\mathrm{Sp}(1)$ is canonically isomorphic to $\mathrm{SU}(2)$. The group G acts on the projective space $P_n(\mathbf{H})$. Let Ω denote the image of the open set $\{x \in \mathbf{H}^{n+1} : [x,x] > 0\}$ under the canonical map $\mathbf{H}^{n+1} \setminus \{0\} \to P_n(\mathbf{H})$. Then G acts transitively on Ω, and the stabilizer of the quaternionic line generated by the vector $(1,0,\dots,0)$ is the group

$$K = \left\{ \begin{bmatrix} u & 0 \\ 0 & U \end{bmatrix} : u \in \mathrm{Sp}(1), U \in \mathrm{Sp}(n) \right\} \equiv \mathrm{Sp}(1) \times \mathrm{Sp}(n)\,.$$

The homogeneous space G/K is called the hyperbolic quaternionic space. K is a maximally compact subgroup of G. G is connected and simply connected.