Proof. Let us consider a coding of the rotation by angle α under the left-closed and right-open partition of the unit circle bounded by all the points of the form $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$; let $\beta_0, \dots, \beta_{p-1}$ denote these consecutive points. The letter associated with the interval $I_k = [\beta_k, \beta_{k+1}[$ has a unique right extension, except when I_k contains points of the form $\{\beta_i - \alpha\}$. Suppose there are $q \geq 2$ points of this form; the associated letter has $q+1$ right extensions. Since there are at most d points of this type, we obtain $p(2) - p(1) \leq d$. We deduce from Theorem 6 that there are at most $3d$ different frequencies for the letters of the coding, i.e., there are at most $3d$ different lengths for the intervals I_k.

REMARK. The start and finish intervals as introduced by Liang in his proof in [37] correspond exactly to the beginning of the branches in the graph of words. Indeed, Liang shows that any interval is associated either with a start point $\{\gamma_i\}$ (i.e., with one extension of a factor having more than one right extension) or with a finish point $\{(n_i - 1)\alpha + \gamma_i\}$ (i.e., with a factor having more than one left extension). Counting the finish and start points defined in [37] (there are $3d$ such points) is equivalent to counting the number of branches in the graph of words.

As in the remark of the previous section, we can consider a coding of the rotation by irrational angle $1-\alpha$ under the partition $\{[\gamma_1, \gamma_2[, \dots, [\gamma_d, \gamma_1[\}$. For such a coding, the $3d$ distance theorem can be rephrased as follows.

THEOREM 20. *The frequencies of the factors of given length $n \geq n^{(1)}$ of a coding of a rotation by irrational angle under a partition in d intervals take at most $3d$ values, where $n^{(1)}$ denotes the connectedness index.*

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