§4. Discriminants

for $A_{N,\beta}$ by simply adding the relation $e = 0$ to the usual presentation of the Temperley-Lieb algebra. For a discussion of other contexts for e, see [MV].

We remark also that it follows from (3.6) (cf. also §5 below) and the theory of cellular algebras that $\mathbf{T}(N)$ is non-semisimple if and only if $N \geq l$. Thus the case $N = l - 1$ is distinguished as the unique one where $\mathbf{T}(N)$ is semisimple, but the Jones form is degenerate.

(3.9) REMARK concerning the Jones (annular) algebras. Since the Jones algebra $\mathbf{J}(n)$ (see (2.10) above) is a quotient of the algebra $\mathbf{T}^a(n)$, any $\mathbf{J}(n)$-module lifts to a $\mathbf{T}^a(n)$-module. The $W_{t,z}(n)$ which correspond to $\mathbf{J}(n)$-modules in this way are those where $z^t = 1$ and $t > 0$ (2.10). Now the conditions $z^2 = q^s$ and $y = zq^{-k}$ (where $s = t + 2k$) of Theorem (3.4) imply (if $t > 0$) that $z^t = 1$ if and only if $y^s = 1$. Hence if $z^t = 1$, the modules $W_{t,z}(n)$ and $W_{s,y}(n)$ of (3.4) may be thought of as $\mathbf{J}(n)$-modules and the map θ_n as a homomorphism of $\mathbf{J}(n)$-modules. If $t = 0$, $z = q$ and the order l of q^2 is finite, then Theorem (3.4) provides a homomorphism $W_{s,y} \to W_{0,q}/M \colon x \mapsto x + M$ where $s = 2l - 2$, $y = q^l (= \pm 1)$ and M is the module defined in (2.9).

§4. DISCRIMINANTS

(4.1) DEFINITION. Throughout this section R denotes the function field $\mathbf{Q}(q)$ and we consider the affine Temperley-Lieb algebras over the ring $R[z, z^{-1}]$ of Laurent polynomials. If $t \leq s$ are non-negative integers of the same parity define

$$[t; s]_{\mathrm{x}} := \begin{bmatrix} s \\ (s-t)/2 \end{bmatrix}_{\mathrm{x}} .$$

The goal of this section is to compute the *discriminant* of the bilinear pairing

$$\langle\ ,\ \rangle_{t,z} \colon W^s_{t,z}(n) \times W^s_{t,z^{-1}}(n) \to R \quad (n \in \mathbf{Z}_{\geq 0}) .$$

This is the determinant of the *gram matrix* $G^s_{t,z}(n)$ with entries $\langle \mu, \nu \rangle_{t,z}$ indexed by pairs of standard monic diagrams $: t \to n$ of rank (strictly) less than $(s-t)/2$. Recall from (2.12) that these diagrams span a $\mathbf{T}(n)$-submodule $W^s_{t,z}(n)$ of $W_{t,z}(n)$ and that these submodules form an increasing filtration of $W_{t,z}(n)$ as s increases. When $n < s$, we write $G_{t,z}(n)$ for this matrix, because it is then independent of s. Similarly define the gram matrix $G^s_{t,0}(n)$ for the pairing $\langle\ ,\ \rangle_{t,0} \colon W^s_{t,0}(n) \times W^s_{t,\infty}(n) \to R$ and let $G_t(n)$ denote the gram matrix of $\langle\ ,\ \rangle_t \colon W_t(n) \times W_t(n) \to R$ with respect to the basis of finite, monic diagrams. We maintain the standard notation $s - t = 2k$.

Recall from (2.13) that there is an idempotent $e_s \in \mathbf{T}(s)$ associated with the trivial representation $W_s(s)$. Define the element $\mathbf{v}_s \in W_{t,z}(s)$ by

$$\mathbf{v}_s := [t;s]_q e_s * \eta^k = \sum_{\substack{\mu:\ t\to s \\ \text{standard}}} e_\mu \mu .$$

Note that by (2.14) $\mathbf{v}_s$ spans the projection of $W_{t,z}(s)$ onto the trivial representation of $\mathbf{T}(s)$. We conjecture, but do not require, that the coefficients (in $R = \mathbf{Q}(q)$) of the Laurent polynomials e_μ in q actually lie in $\mathbf{Z}_{\geq 0}[q, q^{-1}]$.

(4.2) PROPOSITION. *With the notation above,*

(4.2.1) $$\langle \mathbf{v}_s, \eta^k \rangle_{t,z} = \prod_{\substack{t<r\leq s \\ r\equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2}) .$$

Proof. By Lemma (2.11), $e_\mu z^l$ is a polynomial in $R[z^2]$ of degree at most $l = k - |\mu|$. We shall use Theorem (3.4) to compute the value of e_μ when z^2 is specialised to q^s. Taking $n = m = s$ in (3.4.1), we see that $\theta_s(\mathrm{id}_s)$ is annihilated by finite non-monic diagrams $\alpha\colon s \to s$. It follows from (2.14) that $\theta_s(\mathrm{id}_s)$ is a scalar multiple of the specialisation of $\mathbf{v}_s$. The coefficient of η^k in $\theta_s(\mathrm{id}_s)$ is easily checked from the formula (3.4.1) to be 1. Since $e_s * e_s * \eta^k = e_s * \eta^k$, we see that the coefficient of η^k in $e_s * \eta^k$ is also 1, whence after specialisation, we have $[t;s]_q \theta_s(\mathrm{id}_s) = \mathbf{v}_s$. Hence e_μ specialises to

(4.2.2) $$q^i z^{k-|\mu|} h_{P(\mu)}(q) [t;s]_q .$$

Similarly, Corollary (3.5) shows that the coefficient of z^l in e_μ is

(4.2.3) $$h_{F(\mu)}(q)$$

where $h_{F(\mu)}(\mathrm{x})$ is as defined in (3.5).

Now the statement (4.2) will follow by induction on s from the claim:

(4.2.4) $$\eta^* * \mathbf{v}_s = (z^2 - q^s - q^{-s} + z^{-2}) \mathbf{v}_{s-2} .$$

We now proceed to establish (4.2.4), using the observations just made. If α is a finite diagram in $\mathbf{T}(s-2)$, then $\eta \circ \alpha = \beta \circ \eta$ for some finite diagram $\beta\colon s \to s$. If α is not the identity, then β is not monic and so α^* annihilates $\eta^* * \mathbf{v}_s$. It follows from (2.14) that

$$\eta^* * \mathbf{v}_s = \lambda \mathbf{v}_{s-2}$$

for some scalar λ in $R = \mathbf{Q}(q)$.

To determine this scalar, we compute the coefficient of η^{k-1} in $\eta^* * \mathbf{v}_s$ and compare this with the corresponding coefficient $[t; s-2]_q$ in $\mathbf{v}_{s-2}$. In the proof of Theorem (3.4), we enumerated the standard diagrams $\mu: t \to s$ such that $\eta^* \circ \mu = \nu = \eta^{k-1}$. We now compute the contribution of each such μ to the coefficient of η^{k-1} in $\eta^* * \mathbf{v}_s$, just as in the proof of (3.4).

In case 1, $\mu = \eta^k$ and the contribution is

$$-[2]_q[t; s]_q$$

to the coefficient of ν. Cases 2 and 3 do not arise because $a' = 2$ and $b' = s-1$. There are three possibilities that arise in Case 4. Suppose first that μ has nonzero rank, or equivalently that $s-2 > t$; hence $\phi_\mu(1) = 2$ and $\phi_\mu(s) = s-1$ as in the proof of (3.4). It follows that $|\nu| - |\mu| = 2$ and so e_μ has the form $r_2z^2 + r_0 + r_{-2}z^{-2}$ for some r_2, r_0, r_{-2} in $R = \mathbf{Q}(q)$. We have $r_{-2} = r_2$ by symmetry, $r_2 = [t; s-2]_q$ by (4.2.3) and $q^s r_2 + r_0 + q^{-s} r_{-2} = [2]_q[t; s]_q$ by (4.2.2). Thus the contribution of μ is

$$(4.2.5) \qquad (z^2 - q^s - q^{-s} + z^{-2})[t; s-2]_q + [2]_q[t; s]_q .$$

Otherwise we may assume that μ is finite, or equivalently that $t = s-2$. If $t = 0$, then $\phi_\mu(1) = 2 = s$ and the coefficient e_μ has the form $r_1 z + r_{-1}z^{-1}$ for some $r_1, r_{-1} \in R$. We have $r_1 = r_{-1}$ by symmetry and $r_1 = 1$ by (4.2.3). Hence this term contributes $\chi(\tau_0)(z + z^{-1})$ which is equal to the expression in (4.2.5). Next suppose $t = s-2 > 0$. Then either $s \in \mathrm{thr}(\mu)$ and $\phi_\mu(1) = 2$, or $1 \in \mathrm{thr}(\mu)$ and $\phi_\mu(s) = s-1$. In the first case $e_\mu = r_1 z + r_{-1}z^{-1}$ and by symmetry in the second case the coefficient is $r_{-1}z + r_1 z^{-1}$. We have $r_1 = 1$ and $r_{-1} = [s-1]_q$ by (4.2.3). Hence these terms contribute $\chi(\tau)(z + [s-1]_q z^{-1}) + \chi(\tau^{-1})([s-1]_q z + z^{-1})$ which is also equal to the expression (4.2.5).

Each of the three possibilities yields the same contribution (4.2.5), from which it follows that the coefficient of ν in $\eta^* * \mathbf{v}_s$ is $\lambda[t; s-2]_q$ where $\lambda = z^2 - q^s - q^{-s} + z^{-2}$. The claim (4.2.4), and hence the proposition, follows. □

(4.3) COROLLARY. *For non-negative integers $t \le s$ of the same parity, we have the recurrence:*

$$\det G^{s+2}_{t,z}(n) = \det G^s_{t,z}(n) \det G_s(n) \left([t; s]_q^{-1} \prod_{\substack{t<r\le s \\ r\equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2})\right)^{\dim W_s(n)}$$

where $n \in \mathbf{Z}_{\ge 0}$. This, together with the initial condition $\det G^t_{t,z}(n) = 1$ determines $\det G^s_{t,z}(n)$ for any n, s, t.

Proof. Define a basis of $W_{t,z}^{s+2}(n)$ as follows. If $\mu\colon t \to n$ has rank (strictly) less than $k = (s-t)/2$, define $\mathbf{v}_\mu = \mu$. Alternatively, if $\mu\colon t \to n$ has rank k, then (1.9.1) shows that there exists a unique finite monic diagram $\mu'\colon s \to n$ such that $\mu = \mu' \circ \eta^k$; define $\mathbf{v}_\mu = \mu' * \mathbf{v}_s$ and note that $\mathbf{v}_\mu = [t;s]_q \mu$ mod $W_{t,z}^s(n)$. The discriminant of the pairing $\langle -,- \rangle_{t,z}$ with respect to this basis is therefore

$$[t;s]_q^{2 \dim W_s(n)} \det G_{t,z}^{s+2}(n). \tag{4.3.1}$$

We obtain the recurrence above by computing this discriminant in another way.

If $\mu\colon t \to s$ is standard and $\mu \neq \eta^k$, then

$$\langle \mathbf{v}_s, \mu \rangle_{t,z} = 0. \tag{4.3.2}$$

Together with the previous proposition, this implies that for any finite diagram $\alpha\colon s \to s$,

$$\langle \alpha * \mathbf{v}_s, \mathbf{v}_s \rangle_{t,z} = \begin{cases} [t;s]_q \lambda & \text{if } \alpha = \mathrm{id}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3.3}$$

where $\lambda = \langle \mathbf{v}_s, \eta^k \rangle_{t,z}$, which is given explicitly in (4.2.1).

Let $\mu, \nu\colon t \to n$ be standard of rank at most k. If $|\mu| = k$ and $|\nu| < k$, then

$$\langle \mathbf{v}_\mu, \mathbf{v}_\nu \rangle_{t,z} = \langle \mathbf{v}_s, (\mu')^* * \nu \rangle_{t,z} = 0$$

by (4.3.2). If $|\mu| < k$ and $|\nu| < k$, then $\langle \mathbf{v}_\mu, \mathbf{v}_\nu \rangle_{t,z} = \langle \mu, \nu \rangle_{t,z}$. If $|\mu| = |\nu| = k$, then $\langle \mu', \nu' \rangle_t$ is the coefficient of the identity in $\nu'^* \nu'$ and so (4.3.3) shows that

$$\langle \mathbf{v}_\mu, \mathbf{v}_\nu \rangle_{t,z} = [t;s]_q \lambda \langle \mu', \nu' \rangle_t .$$

Therefore the discriminant of the pairing on $W_{t,z}^{s+2}(n)$ with respect to this basis is

$$\det G_{t,z}^s(n) \times \det G_s(n) \left([t;s]_q \lambda \right)^{\dim W_s(n)},$$

which, taking account of (4.3.1) above, completes the proof of (4.3). □

(4.4) COROLLARY. *With the notation above,*

$$\det G_{t,z}^s(n) = \det G_{t,0}^s(n) \prod_{\substack{t<r<s \\ r \equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2})^{\dim W_{r,z}^s(n)}. \tag{4.4.1}$$

Proof. Comparing the coefficients of the highest power of z^{-1} on both sides of (4.3) we see that

$$\det G^{s+2}_{t,0}(n) = \det G^{s}_{t,0}(n) \det G_s(n)[t;s]_q^{-\dim W_s(n)} . \tag{4.4.2}$$

If we write $Q(s) = \det G^s_{t,z}(n)/\det G^s_{t,0}(n)$, then it follows from (4.3) and (4.4.2) that

$$Q(s+2) = Q(s) \prod_{\substack{t<r\le s \\ r\equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2})^{\dim W_s(n)} .$$

This recurrence for $Q(s)$ is easily solved using the fact that $Q(t) = 1$. Taking into account the relation $\dim W^{s+2}_{t,z} = \dim W_t(n) + \dim W_{t+2}(n) + \cdots + \dim W_s(n)$, which is an easy consequence of (2.12.1), the desired equation (4.4.1) follows. □

(4.5) COROLLARY. *With the notation above,*

$$\det G^s_{t,0}(n) = \frac{\det G_{t,0}(n)}{\det G_{s,0}(n)} \prod_{\substack{r\ge s \\ r\equiv t \mod 2}} \left(\frac{[t;r]_q}{[s;r]_q}\right)^{\dim W_r(n)} .$$

Proof. The recurrence (4.4.2) shows that

$$\det G_{t,0}(n) = \det G^s_{t,0}(n) \prod_{\substack{r\ge s \\ r\equiv t \mod 2}} \det G_r(n)[t;r]_q^{-\dim W_r(n)} . \tag{4.5.1}$$

For (4.4.2) to hold for all $s \ge t$, we must take $\det G^s_{s,0}(n)$ to be equal to 1. Hence

$$\prod_{\substack{r\ge s \\ r\equiv t \mod 2}} \det G_r(n) = \det G_{s,0}(n) \prod_{\substack{r\ge s \\ r\equiv t \mod 2}} [s;r]_q^{\dim W_r(n)} .$$

Substituting this into (4.5.1), we obtain the statement. □

(4.6) PROPOSITION. *If $t \le n$ are non-negative integers of the same parity, then*

$$\det G_{t,0}(n) = \pm 1 .$$

Proof. Identify (as above) $\mathbf{n}$ with $u(\{0\}\times\mathbf{n})$. Let $k = (n-t)/2$. Partially order the set of cardinality-k subsets of $\mathbf{n}$ as follows: if $x_1 < x_2 < \cdots < x_k$ and $y_1 < y_2 < \cdots < y_k$ are sequences of elements of $\mathbf{n}$, we say that $\{x_i\} \le \{y_i\}$ if $x_j \le y_j$ for all j in $\mathbf{k}$.

We claim that if $\mu, \nu\colon t \to n$ are standard, then $\langle \mu,\nu\rangle_{t,0} = 0$ unless $\mathrm{rgt}(\mu) \ge \mathrm{lft}(\nu)$. Furthermore if $\mathrm{rgt}(\mu) = \mathrm{lft}(\nu)$, then $\langle\mu,\nu\rangle_{t,0} = 1$. That is, the gram matrix with respect to this pair of ordered bases is triangular with diagonal entries all equal to one, whence its determinant is one. Hence the result will follow from these two claims.

Let $\mu, \nu\colon t \to n$ be standard. Choose graphs for μ and ν with the property that each edge crosses the left side of the fundamental rectangle at most once and recall from section one the construction of a graph for the composition $\sigma = \nu^* \circ \mu$. First suppose that $\langle \mu, \nu \rangle_{t,0} \neq 0$; then $\sigma = \tau^{-|\mu|-|\nu|}\colon t \to t$ since $z = 0$ (cf. (2.11.2)). In this case it is possible to orient the edges of the graphs of μ and ν in such a way that:

(1) Each lower vertex of μ is a source.

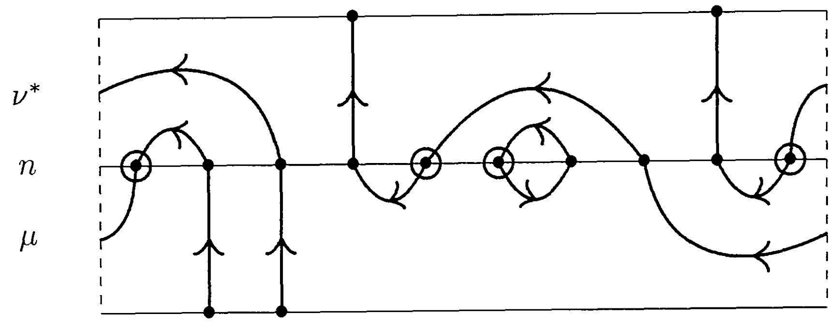
(2) Each lower vertex of ν is a sink.

(3) Each upper vertex $x \in u(\mathbf{Z} \times \mathbf{n})$ is a source (or sink) in precisely one of μ and ν.

(4) Each edge of μ or ν which crosses the left side of the fundamental rectangle is directed from right to left; that is if $x, y \in u(\{0\} \times \mathbf{n})$ are such that $\phi_\mu(x) = V(y)$ (resp. $\phi_\nu(x) = V(y)$) then this edge is directed from y to x in the graph of μ (resp. ν).

To see this, observe that the property (4) implies that when the graphs of μ and ν^* are juxtaposed to form the composition $\nu^* \circ \mu$, the orientations of their edges match, giving an orientation (i.e. linear ordering) to the $\langle \widetilde{\phi_\mu}, \widetilde{\phi_{\nu^*}} \rangle$-orbits on $(\mathbf{Z} \times \mathbf{t}) \amalg (\mathbf{Z} \times \mathbf{n}) \amalg (\mathbf{Z} \times \mathbf{t})$ which are described in the preamble to (1.4). Conversely, such an ordering on these orbits gives an orientation with the required properties. We therefore describe such an ordering or orientation on the orbits which will satisfy the above requirements. If $t = 0$, orient the $g(\sigma) = |\mu| + |\nu|$ infinite loops (see preamble to (1.4) – these correspond to incontractible circuits on the cylinder) from right to left. If $t > 0$, orient each edge of σ ("through string") from the lower vertex to the upper vertex. Note that since $|\nu^* \circ \mu| = |\nu^*| + |\mu|$, all edges of the graphs of μ and ν^* which cross the left side of the fundamental rectangle are included in the edges of the graph of σ, i.e. lie on the through strings of the composite graph. Thus only the contractible (finite) loops which are contained in the fundamental rectangle remain and these may be oriented arbitrarily (say, anti-clockwise). The properties (1) to (4) are clear. Moreover it is also easy to see that if such an orientation exists, then $\sigma = \tau^{-|\mu|-|\nu|}$, since the conditions imply that $|\nu^* \circ \mu| = |\nu^*| + |\mu|$. An example is depicted in the diagram opposite.

Let a denote the i-th element of $\mathrm{rgt}(\mu)$, where $\mathbf{n}$ is identified with $u(\{0\} \times \mathbf{n})$, etc. Now there are at least i sources of the (directed



EXAMPLE

The sources of μ in $\mathbf{n}$ are circled

graph of) μ in the interval $\mathbf{a} \subseteq \mathbf{n}$, because when $y \in \mathrm{rgt}(\mu)$ and $y \leq a$, there is precisely one source in the set $\{y, \phi_\mu(y)\} \cap \mathbf{a}$, by property (4) above. Similarly, let b denote the i-th element of $\mathrm{lft}(\nu)$. Then the above argument shows that there are at least $k - i$ sinks of ν in $\{b+1, b+2, \ldots, n\} \subseteq \mathbf{n}$ and since, by property (2), ν has k sinks in $\mathbf{n}$, there are at most i sinks of ν in $\mathbf{b}$. Moreover if the number of sinks of ν in $\mathbf{b}$ is precisely i, any arc of ν from $b \in \mathbf{n}$ to an element of $\{b+1, b+2, \ldots, n\} \subseteq \mathbf{n}$ must have sink b, otherwise the number of sinks of ν in $\{b+1, b+2, \ldots, n\} \subseteq \mathbf{n}$ would be greater than $k - i$. Now by property (3), a sink of ν is a source of μ. Hence if $b > a$, the number of sources of μ which $\leq b$ is i,so that by the argument just given, μ is a sink of ν, hence a source of μ. But the number of sources of μ which $\leq a$ is $\geq i$. Hence the number of sources of μ which $\leq b$ is at least $i+1$, a contradiction. Hence $b \leq a$ and so $\mathrm{lft}(\nu) \leq \mathrm{rgt}(\mu)$.

Finally, assume that $\mathrm{lft}(\nu) = \mathrm{rgt}(\mu)$. Then in forming the composite $\nu^* \circ \mu$, there are no finite orbits (or contractible loops). For if there were any such orbit, it would be contained in the fundamental rectangle because of the rank condition and hence some element of $u(\{0\} \times \mathbf{n})$ would be in $\mathrm{lft}(\nu) \cap \mathrm{lft}(\mu)$, which is impossible. Hence $\langle \mu, \nu \rangle_{t,0} = 1$. $\square$

As an immediate consequence of (4.4.1) and (4.6), we have

(4.7) COROLLARY. *If $(t, z) \in \Lambda^a$ and $n \in \mathbf{Z}_{\geq 0}$, we have*

$$\det G_{t,z}(n) = \pm \prod_{\substack{r>t \\ r \equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2})^{\dim W_{r,z}(n)} .$$

(4.8) COROLLARY.

(1) *If n is an odd positive integer, then Jones' annular algebra* $\mathbf{J}(n)$ *(with parameter* $\delta = -q - q^{-1}$*) is non-semisimple if and only if there exist distinct odd integers* $s, t \in \mathbf{n}$ *such that* $q^{st} = 1$.

(2) *If n is an even positive integer, then Jones' annular algebra* $\mathbf{J}(n)$ *(with parameter* $\delta = -q - q^{-1}$*) is non-semisimple if and only if* $q^{\frac{n}{2}+1} = 1$ *or there exist distinct even integers* $s, t \in \mathbf{n}$ *such that* $q^{\frac{st}{2}} = 1$.

Proof. By [GL, 3.8] the algebra is semisimple precisely when the bilinear pairing $\langle\ ,\ \rangle_{t,z}$ is non-degenerate on each cell representation (of $\mathbf{J}(n)$); this condition is equivalent to the vanishing of the determinant $\det G_{t,z}(n)$, which by (4.7) immediately yields the stated condition. □

§5. DECOMPOSITION MATRICES

(5.1) THEOREM. *Let R be an algebraically closed field of characteristic zero and q a nonzero element of R. Let $\preceq$ be the weakest partial order on the set Λ^a defined in (2.6) such that $(t,z) \preceq (s,y)$ if (t,z) and (s,y) satisfy the hypotheses of Theorem (3.4) for q or q^{-1}. If $(t,z) \in \Lambda^a$, $n \in \mathbf{Z}_{\geq 0}$ and $(s,y) \in \Lambda^a(n)$, then the multiplicity of the irreducible $\mathbf{T}^a(n)$-module $L_{s,y}(n)$ in the cell representation $W_{t,z}(n)$ of (2.6) is one if $(s,y) \succeq (t,z)$ and zero otherwise.*

Proof. Let R be a field and $q \in R$. Let $p\colon R[\mathrm{y}] \to R$ be the R-algebra homomorphism defined by $\mathrm{y} \mapsto q + q^{-1}$, where y is an indeterminate over R. Suppose W is a free $R[\mathrm{y}]$-module of finite rank with an $R[\mathrm{y}]$-bilinear form $\langle\ ,\ \rangle\colon W \times W \to R[\mathrm{y}]$. If R is regarded as a $R[\mathrm{y}]$-module via the homomorphism p, the free R-module $W_R = R \otimes_{R[\mathrm{y}]} W$ inherits an R-bilinear form $\langle\ ,\ \rangle_R\colon W_R \times W_R \to R$ given by $\langle 1 \otimes x, 1 \otimes y\rangle_R = p(\langle x, y\rangle)$. Choose $R[\mathrm{y}]$-bases B_1 and B_2 of W and let G denote the associated gram matrix of $\langle\ ,\ \rangle$. If this form is nonsingular (i.e. $\det G \neq 0$), then it may be shown that the multiplicity of the polynomial $\mathrm{y} - q - q^{-1}$ in the determinant $\det G$ is greater than or equal to the R-dimension of the radical of $\langle\ ,\ \rangle_R$. In fact if we denote the multiplicity of the polynomial $\mathrm{y} - q - q^{-1}$ in $f \in R[\mathrm{y}]$ by $\mathrm{mult}(f)$, then

$$\mathrm{mult}(\det G) = \sum_{i>0} \dim \mathrm{rad}^i$$

where rad^i denotes the image under $\phi\colon W \to W_R : w \mapsto 1 \otimes w$ of the $R[\mathrm{y}]$-submodule $\{w \in W \mid \langle w, v\rangle \in (\mathrm{y} - q - q^{-1})^i R[\mathrm{y}] \text{ for any } v \in W\}$.