L'Enseignement Mathématique, t. 43 (1997), p. 173–198

POLYGON SPACES AND GRASSMANNIANS

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ABSTRACT. We study the moduli spaces of polygons in $\mathbf{R}^2$ and $\mathbf{R}^3$, identifying them with subquotients of 2-Grassmannians using a symplectic version of the Gel′fand-MacPherson correspondence. We show that the bending flows defined by Kapovich-Millson arise as a reduction of the Gel′fand-Cetlin system on the Grassmannian, and with these determine the pentagon and hexagon spaces up to equivariant symplectomorphism. Other than invocation of Delzant's theorem, our proofs are purely polygon-theoretic in nature.

1. INTRODUCTION

Let ${}^m\widetilde{\mathcal{P}}^k$ be the space of m-gons in $\mathbf{R}^k$ up to translation and positive homotheties (precise definitions in §2). This space comes with several structures: an action of $O(k)$, an action of S_m permuting the edges, and a function $\ell : {}^m\widetilde{\mathcal{P}}^k \longrightarrow \mathbf{R}^m$ taking a polygon ρ to the lengths of its edges (once the perimeter of ρ is fixed). The quotients of ${}^m\widetilde{\mathcal{P}}^k$ by SO_k (or O_k) are the moduli spaces ${}^m\mathcal{P}^k_+$ (respectively, ${}^m\mathcal{P}^k$). Fixing a reflection in $O(k)$ provides an involution on ${}^m\widetilde{\mathcal{P}}^k$ and ${}^m\mathcal{P}^k_+$ whose fixed point sets are ${}^m\widetilde{\mathcal{P}}^{k-1}$ and ${}^m\mathcal{P}^{k-1}$. The goal of this paper is to understand the topology of these various spaces and the geometric structures that they naturally carry when $k = 2$ or 3. They are closely related to more familiar objects (Grassmannians, projective spaces, Hopf bundles, etc.) The spaces ${}^m\mathcal{P}^k(\alpha) := \ell^{-1}(\alpha)$ of polygons with given side-lengths $\alpha \in \mathbf{R}^m$ are of particular interest.

The great miracle occurs when $k = 3$, because $\mathbf{R}^3$ is isomorphic to the space $I\mathbf{H}$ of pure imaginary quaternions, and the 2-sphere in $\mathbf{R}^3$ is Kähler. The tools of symplectic geometry can then be used. Most prominent is a

*) Both authors thank the Fonds National Suisse de la Recherche Scientifique for its support.