0. Introduction

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PARAMETRIZED PLANE CURVES, MINKOWSKI CAUSTICS, MINKOWSKI VERTICES AND CONSERVATIVE LINE FIELDS

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0. INTRODUCTION

Some time ago I made the following observation ([T1]) generalizing the classical 4-vertex theorem. Consider a smooth closed strictly convex parametrized curve $\gamma(t)$ in the oriented affine plane. The acceleration vectors $\gamma''(t)$ (where prime denotes d/dt) generate a smooth line field $l(t)$ along the curve. Assume that these lines rotate in the same sense along γ; analytically this means that $\left[\gamma''(t), \gamma'''(t)\right] \neq 0$ for all t (where $[\ ,\ ]$ denotes the determinant of two vectors).

THEOREM 0.1. *For a generic curve $\gamma(t)$ the envelope of the one-parameter family of lines $l(t)$ has at least 4 cusp singularities.*

If $\gamma(t)$ is a strictly convex curve in the arc-length parameterization then the lines $l(t)$ are perpendicular to γ and their envelope is the caustic of the curve. The singularities of the caustic correspond to the vertices of the curve, i.e., to its curvature extrema. Thus Theorem 0.1 is a generalization of the 4-vertex theorem which asserts that a smooth closed convex plane curve has at least 4 vertices.

The trick used in [T1] to prove the theorem does not explain its relation to concepts of differential geometry, in particular, whether Theorem 0.1 can be interpreted as a 4-vertex theorem.

The purpose of this paper is to provide such an explanation. I will show that Theorem 0.1 is a 4-vertex theorem in Minkowski geometry in the plane associated with the parametrized curve $\gamma(t)$. It will also be seen that the statement of Theorem 0.1 holds even without the assumption $[\gamma''(t), \gamma'''(t)] \neq 0$.

The 4-vertex theorem in the Minkowski plane is by no means new. An equivalent statement can be found in [Bl 1]; see also [Ge, Gu 1, 2, He 1, 2, Su] (note another term, the *relative differential geometry*, classically used to describe the situation).

The point of view in this paper is that of contact geometry which, I believe, clarifies the matter and makes it possible to extend naturally many familiar results from the Euclidean setting to the more general Minkowski and Finsler ones. For an approach to the 4-vertex theorem and related results as theorems of symplectic and contact topology see, e.g., [A 1, A 4].

1. FINSLER METRIC FROM THE CONTACT GEOMETRICAL VIEWPOINT

Finsler geometry describes the propagation of light in an inhomogeneous anisotropic medium. This means that the velocity of light depends on the point and the direction. There are two equivalent descriptions of this process corresponding to the Lagrangian and the Hamiltonian approaches in classical mechanics.

On the one hand, one may study the rays of light, that is, the shortest paths between points. The optical properties of a medium are described by a strictly convex smooth hypersurface, called the *indicatrix*, in the tangent space at each point. The indicatrix consists of the velocity vectors of the propagation of light at a point in all directions. It plays the role of the unit sphere in Riemannian geometry.

The distance $d(x, y)$ between points x and y is the least time it takes light to travel from x to y. If the indicatrices are not centrally symmetric this distance may not be symmetric: $d(x, y) \neq d(y, x)$. However it still satisfies the triangle inequality:

$$d(x, y) + d(y, z) \geq d(x, z).$$

Minkowski geometry is a particular case of Finsler geometry in affine space in which the indicatrices of all points are identified by parallel translations. The rays of light in Minkowski geometry are straight lines.

On the other hand, one may study the wave fronts. The wave front of a point is the hypersurface that consists of points which light can reach from the given point in a fixed time. A wave front is characterized by its contact elements (hyperplanes in the tangent spaces at the points of the front tangent