

## 4. Obtaining Elements of N

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For  $l \in L, v \in \mathbf{Q}^d$  we have  $vF(ln^{-1})^{tr} = va^{-1}nFl^{tr}$  and hence  $(L')^\# = L^\#an^{-1}$ .

Since  $L' \in \pi(L)$  one has  $L' = L^\# \cap a^{-1}L$ . Using this one obtains

$$Lan^{-2} = L'an^{-1} = L^\#an^{-1} \cap Ln^{-1} = (L')^\# \cap L' = L,$$

since  $(L')^\#/L$  is the orthogonal complement of  $L'/L$  in  $L^\#/L$  with respect to the induced quadratic form with values in  $\mathbf{Q}/\mathbf{Z}$ . So  $a^{-1}n^2 \in G$ .

Finally we check that  $n \in N$ . Let  $g \in G$ , then  $n^{-1}gn$  is in  $G = \text{Aut}(F)$  since  $Ln^{-1}gn = L'gn = L'n = L$  and

$$n^{-1}gnFn^{tr}g^{tr}n^{-tr} = n^{-1}agFg^{tr}n^{-tr} = F. \quad \square$$

#### 4. OBTAINING ELEMENTS OF $N$

Now we give examples as to how one may construct elements  $n$  of the normaliser  $N$ . To obtain similarities we are interested in  $n \in N$  of determinant  $\pm p^{d/2}$  for some (squarefree) natural number  $p$  such that  $p^{-1}n^2 \in G$ . The first method is an application of the normaliser principle to the situation (iii) described in Section 2:

**PROPOSITION 4.** *Let  $U \trianglelefteq G$  be a normal subgroup of  $G$  and assume that the commuting algebra  $K := C_{M_d(\mathbf{Q})}(U)$  is isomorphic to a number field. If  $c \in K$  satisfies  $c^2 = p \in \mathbf{Q}^*I_d$ , then  $c$  lies in  $N$ .*

*Proof.* Since  $G$  normalises  $U$ , it acts by conjugation (and hence as Galois automorphisms) on the abelian number field  $K$ . Now let  $c \in K$ , with  $c^2 =: p \in \mathbf{Q}^*I_d$  and  $g \in G$ . Then  $g$  stabilises the subfield  $\mathbf{Q}[c]$  and hence  $g^{-1}cg = \pm c$ , which is equivalent to  $c^{-1}gc = \pm g \in G$ . Therefore  $c \in N$ , since we assumed that  $-I_d \in G$ .  $\square$

The following construction described in [PIN 95] Proposition (II.4) also allows us to find elements of  $N$ .

For  $i = 1, 2$  let  $G_i \leq GL_{d_i}(\mathbf{Q})$  be finite rational irreducible matrix groups with commuting algebras  $A_i \subseteq M_{d_i}(\mathbf{Q})$ . Also let  $Q$  be a maximal common subalgebra of dimension  $z$  of  $A_1$  and  $A_2$ . Let  $d := \frac{d_1 d_2}{z}$  and view the  $G_i$  as subgroups of  $G_1 \otimes_Q G_2 \leq GL_d(\mathbf{Q})$ . If there exist elements  $a_i \in N_{GL_d(\mathbf{Q})}(G_i)$

centralising  $G_j$  and  $a_j$  ( $1 \leq i \neq j \leq 2$ ) and a squarefree natural number  $p \neq 0$  such that  $p^{-1}a_i^2 \in G_i$ , the group

$$G := \langle G_1 \otimes_{\mathbf{Q}} G_2, p^{-1}a_1a_2 \rangle,$$

generated by the elements of  $G_1 \otimes_{\mathbf{Q}} G_2$  and  $p^{-1}a_1a_2$ , is a finite subgroup of  $GL_d(\mathbf{Q})$  containing  $G_1 \otimes_{\mathbf{Q}} G_2$  as a subgroup of index 2.

For  $d \leq 31$  and  $p > 1$  we only need the case where  $a_2$  is an element of the enveloping algebra of  $G_2$ . Then  $G$  is denoted by  $G_1 \otimes_{\mathbf{Q}}^{2(p)} G_2$  (or  $G_1 \boxtimes_{\mathbf{Q}}^{2(p)} G_2$ ) according to whether  $a_1$  is (or is not) a rational linear combination of elements of  $G_1$ .

Using this notation one immediately has the following proposition.

**PROPOSITION 5.** *For  $i = 1, 2$  the matrix  $a_i$  is an element of determinant  $\pm p^{d/2}$  in the normaliser  $N$  of  $G$ .*

A common feature of the situations in Propositions 4 and 5 is that we extend the natural representation of  $G$  to a projective representation which is realisable as a linear representation over a quadratic extension of  $\mathbf{Q}$ .

**PROPOSITION 6.** *Let  $G \trianglelefteq E$  be a supergroup containing  $G$  of index 2. Assume that  $C_{M_d(\mathbf{Q})}(G) \cong \mathbf{Q}$  and that the natural character of  $G$  extends to  $E$  with character field  $\mathbf{Q}[\sqrt{p}]$ , where  $p \in \mathbf{Z}$  is not a square. Then there exists  $n \in N$  of determinant  $\pm p^{d/2}$  with  $p^{-1}n^2 \in G$ .*

*Proof.* By Clifford theory one may extend the natural representation  $\Delta$  of  $G$  to a representation  $\delta_1 \otimes \delta_2 : E \rightarrow (\mathbf{Q}[\sqrt{p}] \otimes M_d(\mathbf{Q}))^*$ , where  $\delta_1$  and  $\delta_2$  are projective representations  $\delta_1(G) = \{1\}$  and  $(\delta_2)|_G = \Delta$ . Let  $e \in E \setminus G$ . Then

$$(\delta_1(e) \otimes \delta_2(e))^2 = \delta_1(e)^2 \otimes \delta_2(e)^2 = 1 \otimes \Delta(e^2),$$

since  $e^2 \in G$ . Therefore  $\delta_1(e)^2 \in \mathbf{Q}$ . Replacing  $\delta_1(e)$  by a suitable rational multiple (and multiplying  $\delta_2(e)$  by the inverse) one may assume that  $\delta_1(e)^2 = p^{-1}$ . Then  $n := \delta_2(e)$  is an element of the normaliser  $N$  with the desired properties.  $\square$

|    | $\text{Aut}(L)$                                                                                       | $\det(L)$                          | $\min(L)$ | $ L_{\min} $          | lattice<br>sparse |
|----|-------------------------------------------------------------------------------------------------------|------------------------------------|-----------|-----------------------|-------------------|
| 4  | $F_4 \tilde{\otimes} F_4 = [2_+^{1+8} \cdot O_8^+(2)]_{16}$                                           | $2^8$                              | 4         | 4320                  | +                 |
| 6  | $[(SL_2(9) \overset{2(3)}{\underset{\infty,3}{\otimes}} SL_2(9)) \cdot 2]_{16}$                       | $3^8$                              | 4         | 720                   | +                 |
| 9  | $[(Sp_4(3) \circ C_3) \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_{16}$                     | $2^8 \cdot 3^8$                    | 6         | 960                   | +                 |
| 14 | $[2 \cdot \text{Alt}_{10}]_{16}$                                                                      | $5^8$                              | 6         | 2400                  | +                 |
| 16 | $[SL_2(5) \overset{2(2)}{\underset{\infty,2}{\boxtimes}} 2^{1+4'} \cdot \text{Alt}_5]_{16}$           | $2^8 \cdot 5^8$                    | 8         | 1200                  | +                 |
| 19 | $[SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} SL_2(9)]_{16}$                               | $3^8 \cdot 5^8$                    | 10        | 1440                  | +                 |
| 21 | $[SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{16}$    | $2^8 \cdot 3^8 \cdot 5^8$          | 12        | 480                   | +                 |
| 25 | $[2 \cdot \text{Alt}_7 \overset{2(3)}{\underset{\sqrt{-7}}{\boxtimes}} \tilde{S}_3]_{16}$             | $3^8 \cdot 7^8$                    | 12        | 1680                  | +                 |
| 26 | $[SL_2(7) \overset{2(3)}{\underset{\sqrt{-7}}{\boxtimes}} \tilde{S}_3]_{16}$                          | $3^8 \cdot 7^8$                    | 10        | 336                   | $p \neq 2$        |
| 3  | $[2 \cdot \text{Co}_1]_{24}$                                                                          | $1^{24}$                           | 4         | 196560                | +                 |
| 6  | $[6 \cdot U_4(3) \cdot 2 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_{24}$                  | $2^{12}$                           | 4         | 3024                  | $p \neq 3$        |
| 16 | $[6 \cdot L_3(4) \cdot 2 \overset{2(2)}{\otimes} D_8]_{24}$                                           | $2^{12} \cdot 3^{12}$              | 8         | 3024 + 7560           | +                 |
| 17 | $[(SL_2(3) \circ C_4) \cdot 2 \overset{2(3)}{\underset{\sqrt{-1}}{\boxtimes}} U_3(3)]_{24}$           | $2^{12} \cdot 3^{12}$              | 8         | 4536 + 6048           | +                 |
| 18 | $A_{24}$                                                                                              | $1^{24}$                           | 2         | 600                   | $p \neq 5$        |
| 22 | $[2 \cdot J_2 \overset{2}{\square} SL_2(5)]_{24}$                                                     | $5^{12}$                           | 8         | 37800                 | +                 |
| 35 | $[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$                                                         | $7^{12}$                           | 8         | 1008 + 3024           | $p \neq 2$        |
| 40 | $[SL_2(13) \overset{2(2)}{\square} SL_2(3)]_{24}$                                                     | $13^{12}$                          | 12        | $2 \cdot 2184 + 8736$ | $p \neq 2$        |
| 42 | $[6 \cdot \text{Alt}_7 : 2]_{24}$                                                                     | $2^{12}$                           | 4         | 3024                  | +                 |
| 43 | $[3 \cdot M_{10} \overset{2(2)}{\underset{\sqrt{-3}}{\boxtimes}} SL_2(3)]_{24}$                       | $2^{12} \cdot 5^{12}$              | 8         | 1080                  | $p \neq 3$        |
| 44 | $[\text{Alt}_5 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{24}$ | $2^{12} \cdot 3^{12} \cdot 5^{12}$ | 16        | $360 + 2 \cdot 720$   | $p \neq 2$        |
| 45 | $[3 \cdot M_{10} \overset{2(2)}{\boxtimes} D_8]_{24}$                                                 | $2^{12} \cdot 3^{12} \cdot 5^{12}$ | 16        | $1080 + 1080$         | +                 |
| 64 | $[SL_2(11) \overset{2(2)}{\underset{\sqrt{-11}}{\boxtimes}} SL_2(3)]_{24}$                            | $2^{12} \cdot 11^{12}$             | 12        | 1320                  | $p \neq 2$        |