

7. Arithmetic intersection theory

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7. ARITHMETIC INTERSECTION THEORY

We recall here the generalization of Arakelov theory to higher dimensions due to Gillet and Soulé. Our main references are [GS1], [GS2] and the exposition in [SABK]. For A an abelian group, $A_{\mathbf{Q}}$ denotes $A \otimes_{\mathbf{Z}} \mathbf{Q}$. Let X be an *arithmetic scheme over $\mathbf{Z}$* , by which we mean a regular scheme, projective and flat over $\text{Spec } \mathbf{Z}$. For $p \geq 0$, let $X^{(p)}$ be the set of integral subschemes of X of codimension p and $Z^p(X)$ be the group of codimension p cycles on X . The p -th Chow group of X : $CH^p(X) := Z^p(X)/R^p(X)$, where $R^p(X)$ is the subgroup of $Z^p(X)$ generated by the cycles $\text{div } f$, $f \in k(x)^*$, $x \in X^{(p-1)}$. Let $CH(X) = \bigoplus_p CH^p(X)$. If X is smooth over $\text{Spec } \mathbf{Z}$, then the methods of [F] can be used to give $CH(X)$ the structure of a commutative ring. In general one has a product structure on $CH(X)_{\mathbf{Q}}$ after tensoring with $\mathbf{Q}$.

Let $D^{p,p}(X(\mathbf{C}))$ denote the space of complex currents of type (p,p) on $X(\mathbf{C})$, and $F_{\infty} : X(\mathbf{C}) \rightarrow X(\mathbf{C})$ the involution induced by complex conjugation. Let $D^{p,p}(X_{\mathbf{R}})$ (resp. $A^{p,p}(X_{\mathbf{R}})$) be the subspace of $D^{p,p}(X(\mathbf{C}))$ (resp. $A^{p,p}(X(\mathbf{C}))$) generated by real currents (resp. forms) T such that $F_{\infty}^* T = (-1)^p T$; denote by $\tilde{D}^{p,p}(X_{\mathbf{R}})$ and $\tilde{A}^{p,p}(X_{\mathbf{R}})$ the respective images in $\tilde{D}^{p,p}(X(\mathbf{C}))$ and $\tilde{A}^{p,p}(X(\mathbf{C}))$.

An *arithmetic cycle* on X of codimension p is a pair (Z, g_Z) in the group $Z^p(X) \oplus \tilde{D}^{p-1,p-1}(X_{\mathbf{R}})$, where g_Z is a *Green current* for $Z(\mathbf{C})$, i.e. a current such that $dd^c g_Z + \delta_{Z(\mathbf{C})}$ is represented by a smooth form. The group of arithmetic cycles is denoted by $\widehat{Z}^p(X)$. If $x \in X^{(p-1)}$ and $f \in k(x)^*$, we let $\widehat{\text{div}} f$ denote the arithmetic cycle $(\text{div } f, [-\log |f_{\mathbf{C}}|^2 \cdot \delta_{x(\mathbf{C})}])$.

The p -th *arithmetic Chow group* of X : $\widehat{CH}^p(X) := \widehat{Z}^p(X)/\widehat{R}^p(X)$, where $\widehat{R}^p(X)$ is the subgroup of $\widehat{Z}^p(X)$ generated by the cycles $\widehat{\text{div}} f$, $f \in k(x)^*$, $x \in X^{(p-1)}$. Let $\widehat{CH}(X) = \bigoplus_p \widehat{CH}^p(X)$.

We have the following canonical morphisms of abelian groups:

$$\begin{aligned} \zeta : \widehat{CH}^p(X) &\longrightarrow CH^p(X), & [(Z, g_Z)] &\longmapsto [Z], \\ \omega : \widehat{CH}^p(X) &\longrightarrow \text{Ker } d \cap \text{Ker } d^c (\subset A^{p,p}(X_{\mathbf{R}})), & [(Z, g_Z)] &\longmapsto dd^c g_Z + \delta_{Z(\mathbf{C})}, \\ a : \tilde{A}^{p-1,p-1}(X_{\mathbf{R}}) &\longrightarrow \widehat{CH}^p(X), & \eta &\longmapsto [(0, \eta)]. \end{aligned}$$

One can define a pairing $\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \rightarrow \widehat{CH}^{p+q}(X)_{\mathbf{Q}}$ which turns $\widehat{CH}(X)_{\mathbf{Q}}$ into a commutative graded unitary $\mathbf{Q}$ -algebra. The maps ζ , ω are $\mathbf{Q}$ -algebra homomorphisms. If X is smooth over $\mathbf{Z}$ one does not have to tensor with $\mathbf{Q}$. The definition of this pairing is difficult; the construction uses the *star product* of Green currents, which in turn relies upon Hironaka's

resolution of singularities to get to the case of divisors. The functor $\widehat{CH}^p(X)$ is contravariant in X , and covariant for proper maps which are smooth on the generic fiber.

Choose a Kähler form ω_0 on $X(\mathbf{C})$ such that $F_\infty^* \omega_0 = -\omega_0$ (this is equivalent to requiring that the corresponding Kähler metric is invariant under F_∞). It is natural to utilize the theory of harmonic forms on X in the study of Green currents on $X(\mathbf{C})$. Following [GS1], we call the pair $\bar{X} = (X, \omega_0)$ an *Arakelov variety*. By the Hodge decomposition theorem, we have $A^{p,p}(X_{\mathbf{R}}) = \mathcal{H}^{p,p}(X_{\mathbf{R}}) \oplus \text{Im } d \oplus \text{Im } d^*$, where $\mathcal{H}^{p,p}(X_{\mathbf{R}}) = \text{Ker } \Delta \subset A^{p,p}(X)$ denotes the space of harmonic (with respect to ω_0) (p, p) forms α on $X(\mathbf{C})$ such that $F_\infty^* \alpha = (-1)^p \alpha$. The subgroup $CH^p(\bar{X}) := \omega^{-1}(\mathcal{H}^{p,p}(X_{\mathbf{R}}))$ of $\widehat{CH}^p(X)$ is called the p -th *Arakelov Chow group* of X . Let $CH(\bar{X}) = \bigoplus_{p \geq 0} CH^p(\bar{X})$. $CH^p(\bar{X})$ is a direct summand of $\widehat{CH}^p(X)$, and there is an exact sequence

$$(11) \quad CH^{p,p-1}(X) \xrightarrow{\rho} \mathcal{H}^{p-1,p-1}(X_{\mathbf{R}}) \xrightarrow{a} CH^p(\bar{X}) \xrightarrow{\zeta} CH^p(X) \longrightarrow 0.$$

In the above sequence the group $CH^{p,p-1}(X)$ is defined as the $E_2^{p,1-p}$ term of a certain spectral sequence used by Quillen to calculate the higher algebraic K -theory of X , and the map ρ coincides with the Beilinson regulator map (cf. [G] and [GS1], 3.5).

If $\mathcal{H}(X_{\mathbf{R}}) = \bigoplus_p \mathcal{H}^{p,p}(X_{\mathbf{R}})$ is a subring of $\bigoplus_p A^{p,p}(X_{\mathbf{R}})$, for example if $X(\mathbf{C})$ is a curve, an abelian variety or a hermitian symmetric space (e.g. a Grassmannian), then $CH(\bar{X})_{\mathbf{Q}}$ is a subring of $\widehat{CH}(X)_{\mathbf{Q}}$. This is not the case in general; for example it fails to be true for the complete flag varieties.

Arakelov [A] introduced the group $CH^1(\bar{X})$, where $\bar{X} = (X, g_0)$ is an arithmetic surface with the metric g_0 on the Riemann surface $X(\mathbf{C})$ given by $\frac{i}{2g} \sum \omega_j \wedge \bar{\omega}_j$. Here g is the genus of $X(\mathbf{C})$ and $\{\omega_j\}$ for $1 \leq j \leq g$ is an orthonormal basis of the space of holomorphic one forms on $X(\mathbf{C})$.

A *hermitian vector bundle* $\bar{E} = (E, h)$ on an arithmetic scheme X is an algebraic vector bundle E on X such that the induced holomorphic vector bundle $E(\mathbf{C})$ on $X(\mathbf{C})$ has a hermitian metric h , which is invariant under complex conjugation, i.e. $F_\infty^*(h) = h$.

To any hermitian vector bundle one can attach characteristic classes $\widehat{\phi}(\bar{E}) \in \widehat{CH}(X)_{\mathbf{Q}}$, for any $\phi \in I(n, \mathbf{Q})$, where $n = \text{rk } E$. For example, we have *arithmetic Chern classes* $\widehat{c}_k(\bar{E}) \in \widehat{CH}^k(X)$. Some basic properties of these classes are:

- (1) $\widehat{c}_0(\bar{E}) = 1$ and $\widehat{c}_p(\bar{E}) = 0$ for $k > \text{rk } E$.
- (2) The form $\omega(\widehat{c}_k(\bar{E})) = c_k(\bar{E}) \in A^{k,k}(X_{\mathbf{R}})$ is the k -th Chern form of the hermitian bundle $\overline{E(\mathbf{C})}$.

- (3) $\zeta(\widehat{c}_k(\bar{E})) = c_k(E) \in CH^k(X)$.
- (4) $f^*\widehat{c}_k(\bar{E}) = \widehat{c}_k(f^*\bar{E})$, for every morphism $f : X \rightarrow Y$ of regular schemes, projective and flat over $\mathbf{Z}$.
- (5) If $\bar{L}$ is a hermitian line bundle, $\widehat{c}_1(\bar{L})$ is the class of $(\operatorname{div}(s), -\log \|s\|^2)$ for any rational section s of L .

Analogous properties are satisfied by $\widehat{\phi}$ for any $\phi \in I(n, \mathbf{Q})$ (see [GS2], Th. 4.1). The most relevant property of these characteristic classes is their behaviour in short exact sequences: if

$$\bar{\mathcal{E}} : 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0$$

is such a sequence of hermitian vector bundles over X , then

$$(12) \quad \widehat{\phi}(\bar{S} \oplus \bar{Q}) - \widehat{\phi}(\bar{E}) = a(\widetilde{\phi}(\bar{\mathcal{E}})).$$

Relation (12) is the main tool for calculating intersection products of classes in $\widehat{CH}(X)$ that come from characteristic classes of vector bundles. Combining it with the results of §4 and §5 gives immediate consequences for such intersections. For example, we have

COROLLARY 4. *Let $\bar{\mathcal{E}} : 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0$ be a short exact sequence of hermitian vector bundles over an arithmetic scheme X . Assume that the metrics on $S(\mathbf{C})$, $Q(\mathbf{C})$ are induced from that on $E(\mathbf{C})$.*

(a) *If $\overline{E(\mathbf{C})}$ is flat, then*

- (1) $\widehat{p}_\lambda(\bar{S} \oplus \bar{Q}) = \widehat{p}_\lambda(\bar{E})$, if λ has length > 1 , and
- (2) $\widehat{p}_k(\bar{S}) + \widehat{p}_k(\bar{Q}) - \widehat{p}_k(\bar{E}) = k\mathcal{H}_{k-1}a(p_{k-1}(\bar{Q}))$, $\forall k \geq 1$,

in the arithmetic Chow group $\widehat{CH}(X)_{\mathbf{Q}}$.

(b) *If $\bar{E} = \bar{L}^{\oplus n}$ for some hermitian line bundle $\bar{L}$ and $\omega = c_1(\overline{L(\mathbf{C})})$, then*

$$\widehat{c}(\bar{S})\widehat{c}(\bar{Q}) - \widehat{c}(\bar{E}) = \sum_{i,j} (-1)^j \binom{n}{i} (\mathcal{H}_n - \mathcal{H}_{n-i} + \mathcal{H}_j) a(\omega^i p_j(\bar{Q})),$$

in the arithmetic Chow group $\widehat{CH}(X)$.