4.3 EX AMPLES OF 1-CONNECTED NON-KÄHLERIAN 3-FOLDS

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where $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$, $h \in H^2(Y, \mathbf{Z})$, *and* $x \in \mathbf{Z}$. *The other topological invariants of* $\mathbf{P}(E)$ *are:*

$$w_2(\mathbf{P}(E)) \equiv \pi^*(w_2(Y) + c_1(E)), p_1(E))$$
$$= \pi^*[c_1(Y)^2 - 2c_2(Y) + c_1(E)^2 - 4c_2(E)],\ b_3(\mathbf{P}(E)) = 0\ .$$

Proof. The Leray-Hirsch theorem identifies the cohomology ring $H^*(\mathbf{P}(E), \mathbf{Z})$ with the ring $H^*(Y, \mathbf{Z})[\xi]/_{<\xi^2 + c_1(E)\cdot\xi + c_2(E)>}$; this determines the cup-form. In order to calculate the characteristic classes one uses the exact sequence $0 \to \mathcal{O}_{\mathbf{P}(E)} \to \pi^*E \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \to T_{\mathbf{P}(E)} \to \pi^*T_Y \to 0$. $b_3(\mathbf{P}(E)) = 0$ follows from $b_1(Y) = 0$ and the Leray-Hirsch theorem.

4.3 Examples of 1-connected non-Kählerian 3-folds

Recall that the Hessian of a symmetric trilinear form $F \in S^3H^\vee$ on a free $\mathbf{Z}$-module H of finite rank was defined as the composition $H_F: H \overset{F^t}{\to} S^2H^\vee \overset{\text{disc}}{\to} \mathbf{Z}$. In terms of coordinates $\xi_1, \dots, \xi_b$ on H it is given by the determinant $\det\left(\frac{\partial^2 f}{\partial\xi_i \partial\xi_j}\right)$, where $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ is the homogeneous cubic polynomial associated with F.

Proposition 16. *Let* F *be a symmetric trilinear form whose Hessian vanishes identically. Then* F *is not realizable as cup-form of a Kählerian 3-fold.*

Proof. Let X be a complex 3-fold with a Kähler metric g. The Kähler class $[\omega_g] \in H^2(X, \mathbf{R})$ defines a multiplication map $\cdot\,[\omega_g]: H^2(X, \mathbf{R}) \to H^4(X, \mathbf{R})$, which is an isomorphism by the Hard Lefschetz Theorem [G/H]. In section 3.1 we have seen that this is not possible if the Hessian of the cup-form vanishes.

Corollary 6. *Cubic forms* $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ *which depend on strictly less than* $b = rk_{\mathbf{Z}}H$ *variables are not realizable as cup-forms of Kählerian 3-folds with* $b_2 = b$.

By considering the Hessian of a cup-form over the reals one obtains further conditions.

Definition 4. *Let* $F \in S^3H^\vee$ *be a symmetric trilinear form on a free* $\mathbf{Z}$*-module of rank* b.

The Hesse cone of F *is the subset* $\mathcal{H}_F \subset H_{\mathbf{R}}$ *defined by* $\mathcal{H}_F := \{h \in H_{\mathbf{R}} \mid (-1)^b \det(F^t(h)) < 0\}$.

The index cone $\mathscr{I}_F$ *of* F *is the subset* $\mathscr{I}_F := \{h \in \mathscr{H}_F \mid F^t(h) \in S^2 H^\vee_{\mathbf{R}}$ *has signature* $(1, -1, \ldots, -1)\}$.

Clearly $\mathscr{I}_F$ is an open subcone of $\mathscr{H}_F$ which coincides with $\mathscr{H}_F$ iff $b \leqslant 2$.

THEOREM 5. *Let* $F_X \in S^3 H^2(X, \mathbf{Z})^\vee$ *be the cup-form of a smooth projective 3-fold with* $h^{0,2}(X) = 0$. *Then* F_X *has a non-empty index cone.*

Proof. Let $h \in H^2(X, \mathbf{Z})$ be the dual class of a hyperplane section Y in some projective embedding. The inclusion $i: Y \hookrightarrow X$ induces a monomorphism $i^*: H^2(X, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ by the weak Lefschetz theorem. The symmetric bilinear form $F_X^t(h) \in S^2 H^2(X, \mathbf{Z})^\vee$ is simply the pull-back of the cup-form of Y under the inclusion i^*; it is therefore non-degenerate by the Hard Lefschetz theorem [L]. Applying the Hodge index theorem to Y we see that the real bilinear form $F_X^t(h) \in S^2 H^2(X, \mathbf{R})^\vee$ must have one positive and $b - 1$ negative eigenvalues. In other words: $h \in I_{F_X}$.

REMARK 13. This result has two applications: it provides topological 'upper bounds' for the ample cone of a projective 3-fold with $h^{0,2} = 0$, and if gives further restrictions on symmetric trilinear forms to be realizable as cup-forms of projective 3-folds with $h^{0,2} = 0$ if $b \geqslant 4$.

These applications will be discussed in section 5.

We will now describe examples of 1-connected, non-Kählerian, complex 3-folds and determine their topological structure.

EXAMPLE 10 (Calabi-Eckmann). E. Calabi and B. Eckmann have defined complex structures X_τ, depending on a parameter τ, on the product $S^3 \times S^3$ [C/E]. Their manifolds are principal fiber bundles over $\mathbf{P}^1 \times \mathbf{P}^1$ whose fiber and structure group is the elliptic curve $E_\tau = \mathbf{C}/_{\mathbf{Z} \oplus \mathbf{Z}\tau}$, $\mathrm{Im}(\tau) > 0$.

The Calabi-Eckmann manifolds are homogeneous, non-Kählerian 3-folds of algebraic dimension 2.

EXAMPLE 11 (Maeda). H. Maeda has generalized the Calabi-Eckmann construction. He constructed fiber bundles X_τ' over Hirzebruch surfaces $\mathbf{F}_n$, $n \geqslant 0$, whose fiber and structure group are an elliptic curve E_τ and $\mathrm{Aut}(E_\tau)$ respectively [M]. X_τ' is again diffeomorphic to $S^3 \times S^3$, and therefore non-Kählerian. Maeda's manifolds X_τ' are homogeneous if and only if $n = 0$ in which case they are Calabi-Eckmann 3-folds.

The Calabi-Eckmann construction can also be generalized in the following way:

Let $S^2 \tilde{\times} S^4$ be the non-trivial S^4-bundle over S^2, i.e. $S^2 \tilde{\times} S^4$ is the unique 1-connected, closed, oriented, differentiable 6-manifold with $H_2(S^2 \tilde{\times} S^4, \mathbf{Z}) \cong \mathbf{Z}$ and $b_3 = 0$, whose cup-form and Pontrjagin class vanish, but whose Stiefel-Whitney class w_2 is non-zero.

THEOREM 6. *For any integer* $b \geqslant 0$ *there exist compact complex 3-folds* X_b, *and* $X_b^{\sim}$ *if* $b \geqslant 1$, *which are homeomorphic to* $\#_b S^2 \times S^4 \#_{b+1} S^3 \times S^3$, *and* $S^2 \tilde{\times} S^4 \#_{b-1} S^2 \times S^4 \#_{b+1} S^3 \times S^3$.

Proof. Let Y be a 1-connected, compact complex surface with $p_g(Y) = 0$ and $b_2(Y) \geqslant 2$, and let $E = \mathbf{C}/_\Gamma$ be the elliptic curve associated to the lattice $\Gamma \subset \mathbf{C}$. We want to construct the required 3-folds as total spaces of principal E-bundles over Y. Let $c: H_2(Y, \mathbf{Z}) \to \Gamma$ be an arbitrary epimorphism. The corresponding cohomology class $c \in H^2(Y, \Gamma)$ defines a topological principal bundle over Y with fiber and structure group $E = \mathbf{C}/_\Gamma$ as follows immediately from the identification of the classifying space $BE \simeq K(\Gamma, 2)$.

Let $\mathscr{O}_Y(E)$ be the sheaf of germs of holomorphic maps from Y to E. We have a short exact sequence $0 \to \Gamma \to \mathscr{O}_Y \to \mathscr{O}_Y(E) \to 0$ and a corresponding exact cohomology sequence

$$\to H^1(Y, \mathscr{O}_Y) \to H^1(Y, \mathscr{O}_Y(E)) \xrightarrow{\delta} H^2(Y, \Gamma) \to H^2(Y, \mathscr{O}_Y) \to$$

By our assumptions δ is an isomorphism, so that every topological principal E-bundle admits a holomorphic structure. Let X be the total space of such a bundle corresponding to a surjective map $c: H_2(Y, \mathbf{Z}) \to \Gamma$. The homotopy sequence of the fibration $p: X \to Y$ yields the sequence

$$0 \to \pi_2(X) \xrightarrow{p_*} \pi_2(Y) \to \pi_1(E) \to \pi_1(X) \xrightarrow{p_*} \pi_1(Y) \to 0 .$$

Since Y is 1-connected, $\pi_2(Y)$ can be identified with $H_2(Y, \mathbf{Z})$, and then the boundary map $\pi_2(Y) \to \pi_1(E)$ becomes the characteristic map $c: H_2(Y, \mathbf{Z}) \to \Gamma$ of the bundle. This implies $\pi_1(X) = \{1\}$, whereas $H_2(X, \mathbf{Z})$ is given by: $0 \to H_2(X, \mathbf{Z}) \xrightarrow{p_*} H_2(Y, \mathbf{Z}) \xrightarrow{s} \Gamma \to 0$.

In particular, $H_2(X, \mathbf{Z})$ is free as a submodule of $H_2(Y, \mathbf{Z})$, and by dualizing the last sequence we obtain an identification (via p^*)

$$H^2(X, \mathbf{Z}) = H^2(Y, \mathbf{Z})/_{\Gamma^\vee} .$$

The cup-form F_X of X is therefore trivial. In order to calculate $p_1(X)$ and $w_2(X)$, we use the exact sequence of tangent sheaves: $0 \to T_{X/Y} \to T_X$

$\to p^* T_Y \to 0$. Since $T_{X/Y}$ is a trivial bundle, the characteristic classes of X are simply the pullbacks of the corresponding classes of Y. But the map $p^* : H^4(Y, \mathbf{Z}) \to H^4(X, \mathbf{Z})$ is zero, since $< p^*(\varepsilon) \cup p^*(\alpha), [X] >$ $= < \varepsilon \cup \alpha, p_*[X] > = 0$ for all classes $\varepsilon \in H^4(Y, \mathbf{Z})$, and $\alpha \in H^2(Y, \mathbf{Z})$.

Thus $p_1(X) = 0$, and $w_2(X)$ is the residue of $w_2(Y) \in H^2(Y, \mathbf{Z}_{/2})$ modulo $\Gamma^\vee/_{2\Gamma^\vee}$.

The Euler characteristic of X is zero, so that from $b_2(X) = b_2(Y) - 2$ we find $b_3(X) = 2(b_2(Y) - 1)$. The system of invariants associated to the manifold X is therefore given by

$$(b_2(Y) - 1, H^2(Y, \mathbf{Z})/_{\Gamma^\vee}, w_2(Y) \,(\mathrm{mod}\, \Gamma^\vee/_{2\Gamma^\vee}), 0, 0, 0) ,$$

i.e. X is diffeomorphic to

$$\#_{b_2(Y)-2} S^2 \times S^4 \#_{b_2(Y)-1} S^3 \times S^3 \text{ if } w_2(Y) \in \Gamma^\vee/_{2\Gamma^\vee} ,$$

and to $S^2 \tilde{\times} S^4 \#_{b_2(Y)-3} S^2 \times S^4 \#_{b_2(Y)-1} S^3 \times S^3$ if $b_2(Y) \geqslant 3$, and $w_2(Y) \notin \Gamma^\vee/_{2\Gamma^\vee}$.

EXAMPLE 12 (Kato). In the two papers [K1], [K2] M. Kato studies the class of compact, complex 3-folds X containing smooth rational curves with neighborhoods biholomorphic to those of projective lines in $\mathbf{P}^3$. On this class of 3-folds, called class L, he defines a semi-group structure $+$ with neutral element $\mathbf{P}^3$.

Kato's connecting operation $+$ is defined by removing 'lines' $L_i \subset X_i$ from 3-folds X_i, $i = 1, 2$, and by identifying the complements $X_i \backslash L_i$ along open sets $U_i \backslash L_i$ obtained from suitable neighborhoods $U_i \subset X_i$.

Starting with a certain elliptic fiber space X_1 over the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ in a point, he constructs a sequence of 3-folds $X_n := X_1 + X_{n-1}, n \geqslant 2$. The 3-folds X_n are 1-connected spin-manifolds with $H_2(X_n, \mathbf{Z}) = \mathbf{Z}$. Their cup-forms F_{X_n}, and their Pontrjagin classes $p_1(X_n)$ are in terms of a (normalized) generator $e_n \in H^2(X_n, \mathbf{Z})$ and its dual class $\varepsilon_n \in H^4(X_n, \mathbf{Z})$ given by $F_{X_n}(xe_n) = (n-1)x^3$, and $p_1(X_n) = 4(n-1)\varepsilon_n$ $(\varepsilon_n(e_n) = 1)$. The third Betti-number of X_n is $4n$.

In particular, X_1 is diffeomorphic to $S^2 \times S^4 \#_2 S^3 \times S^3$, and X_2 is diffeomorphic to $\mathbf{P}^3 \#_4 S^3 \times S^3$. It is interesting to note that the Chern-numbers $c_1^3, c_1 c_2$ of the X_n are $c_1^3 = 64(1-n)$, $c_1 c_2 = 24(1-n)$, i.e. they satisfy $8c_1 c_2 = 3c_1^3$. For projective manifolds of general type this equality is characteristic for ball quotients [Y].

EXAMPLE 13 (Twistor spaces). Let $p: Z \to M$ be the twistor fibration of a closed, oriented Riemannian 4-manifold (M, g). Z carries a natural almost complex structure which is integrable if and only if g is self-dual [A/H/S].

Examples of 1-connected 4-manifolds which admit self-dual structures are $S^4, \sharp_n \mathbf{P}^2$, and $K3$-surfaces.

The total spaces of their twistor fibrations are 1-connected complex 3-folds which may be Moishezon for S^4 and $\sharp_n \mathbf{P}^2$ [C], but which are usually non-Kähler [Hi]. We leave it to the reader to calculate the topological invariants of these 3-folds. There is an interesting relation between Twistor spaces of connected sums and Kato's connection operation $+$ for class L manifolds [K2], [D/F].

EXAMPLE 14 (Oguiso). In a recent preprint [O1] K. Oguiso constructs examples of 1-connected, Moishezon Calabi-Yau 3-folds with very interesting cup-forms. He proves that for every integer $d \geqslant 1$ there exists a smooth complete intersection X'_d of type $(2, 4)$ in $\mathbf{P}^5$ which contains a non-singular rational curve C_d of degree d with normal bundle $N_{C_d/X_d} = \mathscr{O}_{C_d}(-1)^{\oplus 2}$.

The 3-fold X'_d can now be flopped along C_d, i.e. C_d can be blown up to $\mathbf{P}(N_{C_d/X_d}) \cong \mathbf{P}^1 \times \mathbf{P}^1$, and then 'blown down in the other direction'. The resulting 3-fold X_d is a 1-connected Moishezon manifold with trivial canonical bundle and cup-form F_{X_d} given by $F_{X_d}(xe_d) = (d^3 - 8)x^3$. Here $e_d \in H^2(X_d, \mathbf{Z})$ is the normalized generator corresponding to the strict transform of the negative of a hyperplane section of X'_d. The Pontrjagin class of X_d is $p_1(X_d) = (112 + 4d)\varepsilon_d$ where $\varepsilon_d \in H^4(X_d, \mathbf{Z})$ denotes the generator with $\varepsilon_d(e_d) = 1$. Since the Euler-number does not change under a flop we have $b_3(X_d) = 180$ for every d.

5. COMPLEX 3-FOLDS WITH SMALL b_2

In this section we investigate the following natural problem: Which cubic forms can be realized as cup-forms of compact complex 3-folds? For small b_2 something can be said: Any core of a 1-connected, closed, oriented differentiable 6-manifold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ is homotopy equivalent to the core of a 1-connected complex 3-fold. In the case $b_2 = 2$, at least every discriminant Δ is realizable by a complex manifold. If $b_2 = 3$ we can realize all types of complex cubics with one exception, the union of a smooth conic and a tangent line. In addition to these realization results we prove a finiteness