3.3 Arithmetical aspects

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semi-stable for $\alpha \neq 0$ with Aronhold invariants $S = 4\alpha^4$, $T = 8\alpha^6$. The fiber of π over 0 contains 6 orbits with normal forms

$$Y^2Z - X^3,\ Y(X^2 - YZ),\ XY(X + Y),\ X^2Y,\ X^3,$$

and 0, of which the last 4 types have vanishing Hessians. For more details we refer to H. Kraft's book [Kr].

REMARK 8. The natural $\mathbf{C}^*$-action $f \to \lambda \cdot f$ on cubic forms induces a weighted action on the GIT quotient $S^3H_{\mathbf{C}}^{\vee}/_{SL(H_{\mathbf{C}})}$, $\lambda \cdot (S, T) = (\lambda^4 S, \lambda^6 T)$. The associated weighted projective space $\mathbf{P}^1(4, 6)$ with homogeneous coordinates $<S, T>$ is the good quotient for semi-stable plane cubic curves. Its affine part $\mathbf{P}^1 \setminus (\Delta)_0$ is the moduli space of genus-1 curves. The $PGL(H_{\mathbf{C}})$-invariant $J := \frac{S^3}{\Delta}$ gives the J-invariant of the corresponding curve.

3.3 ARITHMETICAL ASPECTS

Let $c\colon S^3H^{\vee}/_{SL(H)} \to S^3H_{\mathbf{C}}^{\vee}/_{SL(H)}$ be the map which associates to the $SL(H)$-orbit $<F>$ of a symmetric trilinear form $F \in S^3H^{\vee}$ the $SL(H_{\mathbf{C}})$-orbit $<F>_{\mathbf{C}}$ of its complexification. The c-fiber over $<F>_{\mathbf{C}}$ can be identified with the subset $(SL(H_{\mathbf{C}}) \cdot F \cap S^3H^{\vee})/_{SL(H)}$ of $S^3H^{\vee}/_{SL(H)}$. C. Jordan has shown that these subsets are finite provided the cubic form $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ associated to F has a non-vanishing discriminant [J1]. Jordan's original proof, which is only two pages long, is somewhat hard to follow. The following theorem of A. Borel and Harish-Chandra provides, however, a vast generalization of Jordan's finiteness result:

THEOREM 3 (Borel/Harish-Chandra). *Let G be a reductive $\mathbf{Q}$-group, $\Gamma \subset G$ an arithmetic subgroup, $\xi\colon G \to GL(V)$ a $\mathbf{Q}$-morphism, and $L \subset V$ a Γ-invariant sublattice of $V_{\mathbf{Q}}$. If $v \in V$ has a closed G-orbit in V, then $G_v \cap L/_{\Gamma}$ is a finite set.*

Proof. [B].

COROLLARY 4. *Let $F \in S^3H^{\vee}$ be a symmetric trilinear form on H. If the $SL(H_{\mathbf{C}})$-orbit of F in $S^3H_{\mathbf{C}}^{\vee}$ is closed, then the fiber $c^{-1}(<F>_{\mathbf{C}})$ over $<F>_{\mathbf{C}}$ is finite.*

To check whether a $SL(H_{\mathbf{C}})$-orbit $SL(H_{\mathbf{C}}) \cdot F$ is closed in $S^3H_{\mathbf{C}}^{\vee}$, one has a generalization of the Hilbert-criterion [Kr]: $SL(H_{\mathbf{C}}) \cdot F$ is closed in $S^3H_{\mathbf{C}}^{\vee}$ if and only if for every 1-parameter subgroup $\lambda\colon \mathbf{C}^* \to SL(H_{\mathbf{C}})$, for which $lim_{t \to 0} \lambda(t) \cdot F$ exists in $S^3 H_{\mathbf{C}}^{\vee}$, this limit is already contained in $SL(H_{\mathbf{C}}) \cdot F$. A sufficient condition for $SL(H_{\mathbf{C}}) \cdot F$ to be closed follows from another result of C. Jordan [J2]:

THEOREM 4 (Jordan). *Let* $f \in \mathbf{C}[H_{\mathbf{C}}]_d$ *be a homogeneous polynomial of degree* $d \geqslant 3$. *If its discriminant* $\Delta(f)$ *is non-zero, then* f *has a finite isotropy group* $SL(H_{\mathbf{C}})_f$.

COROLLARY 5. *Let* $F \in S^3 H^{\vee}$ *be a form whose associated cubic polynomial* $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ *has* $\Delta(f) \neq 0$. *Then* $SL(H_{\mathbf{C}}) \cdot F$ *is closed in* $S^3 H_{\mathbf{C}}^{\vee}$.

Proof. Standard arguments, cf. [Bo].

REMARK 9. Closedness of the $SL(H_{\mathbf{C}})$-orbit of F is only a sufficient condition for the finiteness of the fiber $c^{-1}(<F>_{\mathbf{C}})$. There exist other finiteness theorems for special types of forms, like e.g. forms which decompose into linear factors.

Some of these results are surveyed in Volume III of L. Dickson's book [D].

We say that two forms $F, F' \in S^3 H^{\vee}$ belong to the same (proper) equivalence class if they lie in the same $(SL(H)\text{-})GL(H)$-orbit. The group $\mathbf{Z}_{/2} = GL(H)/_{SL(H)}$ acts on the set $S^3 H^{\vee}/_{SL(H)}$ of proper classes, and the quotient becomes the orbit space $S^3 H^{\vee}/_{GL(H)}$.

The $\mathbf{Z}_{/2}$-action is not free in general, but for finiteness properties this plays no rôle.

EXAMPLE 7. Binary cubics

Let H be a free $\mathbf{Z}$-module of rank $b = 2$. There exist only finitely many classes of symmetric trilinear forms $F \in S^3 H^{\vee}$ with a given non-zero discriminant Δ. Of course, Δ must be integral, and a square modulo 4, in order to be realizable by an integral form. For some small values of $\Delta \neq 0$ the number of classes is known. Results in this direction go back to a paper by F. Arndt [A]; his tables have been rearranged by A. Cayley [Cay]. It should certainly be possible to go much further using modern computers.

EXAMPLE 8. Ternary cubics

Let H be a free $\mathbf{Z}$-module of rank 3 with coordinates X, Y, Z. The cubic polynomials with closed $SL(H_{\mathbf{C}})$-orbits are the non-singular cubics, and the polynomials in the orbits of $6\alpha XYZ$ for all $\alpha \in \mathbf{C}$.

The number of integral classes in these orbits is therefore finite. We have, however, an even stronger finiteness theorem for stable ternary cubics:

PROPOSITION 7. *Let H be a free $\mathbf{Z}$-module of rank 3. There exist only finitely many classes of symmetric trilinear forms $F \in S^3 H^\vee$ with a fixed discriminant $\Delta \neq 0$.*

Proof. In terms of Arnhold's invariants S and T, Δ is given by $\Delta = S^3 - T^2$. By a theorem of C. Siegel [Si], the diophantine equation $S^3 - T^2 = \Delta$ has only finitely many integral solution (S, T) for any integer $\Delta \neq 0$. For each of these solutions the corresponding point in $S^3 H_{\mathbf{C}}^\vee / _{SL(H_{\mathbf{C}})}$ lies outside of the discriminant curve, so that the π-fiber over it is a closed $SL(H_{\mathbf{C}})$-orbit. The finiteness of the class number then follows from the Borel/Harish-Chandra theorem.

A famous special case of Siegel's theorem is Bachet's equation $S^3 - T^2 = 2$; it has only the two obvious solutions $(3, \pm 5)$.

REMARK 10. To get finiteness results for ternary cubic forms it is not sufficient to fix the J-invariant (instead of the discriminant): The forms $f_m = X^3 + XZ^2 + Z^3 + mY^2 Z$, $m \in \mathbf{Z} \setminus \{0\}$, all have the same J-invariant, but they are not equivalent, even over $\mathbf{Q}$, since they have bad reduction at different primes $p \mid m$.

4. INVARIANTS OF COMPLEX 3-FOLDS

In this section we begin to investigate the topology of 1-connected, compact, complex 3-folds. After a brief discussion of the possible systems of Chern numbers of almost complex 6-manifolds, we study the behaviour of the topological invariants of complex 3-folds under certain standard constructions, like e.g. branched coverings, or blow-ups of points and curves. Then we describe some interesting examples of 1-connected, non-Kählerian 3-folds, including a new construction method which generalizes the Calabi-Eckmann manifolds. These examples will be needed in the next section in order to realize complex types of cubic forms as cup-forms of complex 3-folds.

4.1 CHERN NUMBERS OF ALMOST COMPLEX STRUCTURES

Let X be a closed, oriented, 6-dimensional differentiable manifold. The tangent bundle of X is induced by a classifying map $t_X : X \to BSO(6)$ which is unique up to homotopy. By an almost complex structure on X we mean the homotopy class $[\tilde{t}_X]$ of a lifting $\tilde{t}_X : X \to BU(3)$ of t_X to $BU(3)$.